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1st Volume dedicated to F. BORCEUX

FOREWORD

In October 2008, an International Conference was organized at the Royal Flemish Academy of Belgium in Brussels to celebrate the 60^{th} birthdays of Francis Borceux and Dominique Bourn. After that, we decided to honour their important contribution to category theory by devoting a volume of the "Cahiers" to Francis Borceux, and a volume of TAC to Dominique Bourn. This is the first issue of the volume for Francis.

Francis obtained his Thesis [2] in 1972, under the direction of René Lavendhomme. Assistant in Leuven from 1970 to 1973, he was appointed Chargé de cours (1973-1980) and Professeur (since 1980) in Louvain-la-Neuve, where he directed 12 theses and, from 1996 to 2001, acted as Dean of the Faculty of Sciences.

We are very grateful to Francis for his role as a highly original researcher in category theory and as one of the most active members of the community in the diffusion of category research for more than 35 years. He regularly organized beautiful category conferences (e.g., the Haute-Bodeux meetings) and summer schools in the Ardennes, where he always took great care to invite not only leading category theorists from all over the world, but also young research students who were given the opportunity to talk about their research. Many generations of mathematicians have greatly benefited from both his scientific guidance and his generous support.

He published, alone or in collaboration, 11 research monographs and textbooks; of particular value is his three-volume comprehensive *Handbook of Categorical Algebra*. He is also an author or co-author of more than eighty research papers, as indicated by his list of publications here below.

The Editors of the volume:
J. Adamek, A.C. Ehresmann, M. Gran, G. Janelidze and R. Kieboom

List of Borceux' publications

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CAHIERS DE TOPOLOGIE ET *Vol. L-3 (2009)*
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

COVERING MORPHISMS AND NORMAL EXTENSIONS IN GALOIS STRUCTURES ASSOCIATED WITH TORSION THEORIES

*by Marino GRAN and George JANELIDZE**

Dedicated to Francis Borceux on the occasion of his 60th birthday

ABSTRACT. Nous étudions les revêtements et les extensions normales relatives aux structures galoisiennes munies de ce que nous appelons *foncteurs test*. Ces foncteurs apparaissent naturellement dans les structures galoisiennes associées aux théories de torsion dans les catégories homologiques. Sous des hypothèses additionnelles appropriées, tout morphisme à noyau sans torsion est un revêtement, et tout revêtement est une extension normale, pourvu qu'il soit un morphisme de descente effective. Nos contre-exemples, qui montrent l'importance de ces conditions supplémentaires, sont semi-abéliens, et proviennent de la théorie des groupes, en faisant intervenir des produits semi-directs de groupes cycliques. Nous comparons nos nouveaux résultats avec ceux connus pour les revêtements localement semi-simples et pour les extensions centrales généralisées.

ABSTRACT. We study covering morphisms and normal extensions with respect to Galois structures equipped with what we call test functors. These test functors naturally occur in Galois structures associated with torsion theories in homological categories. Under suitable additional conditions, every morphism with a torsion free kernel is a covering, and every covering is a normal extension whenever it is an effective descent morphism. Our counter-examples showing the relevance of those additional conditions are semi-abelian, and moreover, group-theoretic, involving semidirect products of cyclic groups. We also briefly compare our new results with what is known for the so-called locally semi-simple coverings and for generalized central extensions.

Introduction

The purpose of *categorical Galois theory* is to study covering morphisms in general categories defined with respect to so-called Galois structures,

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which sometimes are merely abstract semi-left-exact reflections in the sense of [CHK] (see e.g. [J1], [BorJ], [J2]). Apart from classical cases, where the covering morphisms become (quasi-) separable algebras over commutative rings, ordinary covering maps of locally connected topological spaces, étale coverings of schemes in algebraic geometry, etc., there are non-trivial examples far away from commutative algebra and algebraic topology and geometry, such as generalized central extensions in congruence modular varieties of universal algebras. Another "non-classical" case is the Galois theory associated with a torsion theory; it was briefly examined in [CJKP] in the abelian case, and then in [GR] for non-abelian torsion theories in the sense of [BG]. While being Galois theory of the torsion-free reflection, it also substantially uses the torsion coreflection, and clearly suggests considering a more general situation of an abstract Galois structure equipped with what we call a *test functor* because such a functor T is required to "test" trivial covering morphisms via the equivalence

(A,f) is a trivial covering $\Leftrightarrow T(f)$ is an isomorphism.

As explained in Section 2 below, in the case of a torsion theory this condition essentially follows from *Bourn protomodularity*.

The purpose of the present paper is to continue the study of Galois theories associated with torsion theories, and, specifically, to prove/explain/clarify the following:

• Under suitable additional conditions, every morphism with a torsion free kernel is a covering, and every covering is a normal extension whenever it is a monadic extension (=an effective descent morphism).

• There are simple (counter-)examples showing the relevance of those additional conditions. The ones we consider have varieties of groups as their ground categories, and all groups used in the covering morphisms we construct are nothing but semidirect products of cyclic groups.

• The ground structure needed to obtain our main results is far more general than a torsion theory: it is a *finitely complete admissible Galois* *structure* equipped with the above-mentioned test functor: in fact it is a new notion we introduce.

• Coverings defined via torsion theories are to be compared with central extension defined via Birkhoff subcategories [JK] (see also [J2], [G2], and references there), and with locally semisimple coverings in the sense of [JMT1].

The paper is divided into six sections as follows:

1. Coverings and normal extensions in general categories. It recalls basic notions of categorical Galois theory; the next sections freely use them and their simple properties.

2. Covering morphisms under the presence of a test functor. Test functors are introduced and our main results are presented as simple propositions on a Galois structure equipped with a test functor.

3. Galois structures of torsion theories. The main results are translated into the context of a torsion theory.

Sections 4 and 5 present our examples, and Section 6 makes brief comparisons with central extensions and locally semisimple coverings.

1. Coverings and normal extensions in general categories

A *finitely complete admissible Galois structure* $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ (as defined in [J2], slightly differently from the original definition in [J1]) on a category **C** consists of an adjunction

$$(I,H,\eta,\varepsilon) : \mathbf{C} \to \mathbf{X} \qquad (1.1)$$

between categories with finite limits, and two classes **F** and **Φ** of morphisms in **C** and **X** respectively, whose elements are called fibrations; the following conditions on fibrations are required:

• the classes of fibrations are pullback stable;

• the classes of fibrations are closed under composition and contain all isomorphisms;

• the functors I and H preserve fibrations;

• for every object C in $\mathbf{C}$ and every fibration $\varphi : X \to I(C)$ in $\mathbf{X}$, the composite

$$I(C \times_{HI(C)} H(X)) \to IH(X) \to X \tag{1.2}$$

of canonical morphisms is an isomorphism. This last condition is called *admissibility*.

We assume such a structure to be fixed, and, for an arbitrary object C in $\mathbf{C}$, write

$$(I^C, H^C, \eta^C, \varepsilon^C) : \mathbf{F}(C) \to \mathbf{\Phi}(I(C))$$

for the usual induced adjunction, in which:

• $\mathbf{F}(C)$ is the full subcategory in $(\mathbf{C}\downarrow C)$ with objects all (A,f) in $(\mathbf{C}\downarrow C)$, in which $f : A \to C$ is a fibration;

• similarly $\mathbf{\Phi}(I(C))$ is the full subcategory in $(\mathbf{X}\downarrow I(C))$ with objects all (X,φ) in $(\mathbf{X}\downarrow I(C))$, in which $\varphi : X \to I(C)$ is a fibration;

• $I^C(A,f) = (I(A),I(f))$, $H^C(X,\varphi) = (C \times_{HI(C)} H(X), pr_1)$, and η^C and ε^C are defined accordingly; in particular $(\varepsilon^C)_{(X,\varphi)}$ is determined by the composite (1.2) and so the admissibility condition simply says that ε^C is an isomorphism for each C in $\mathbf{C}$.

Let us recall (e.g. again from [J2]):

Definition 1.1. (a) For a fibration $p : E \to B$ in **C**, the object (E,p) in $\mathbf{F}(B)$ is said to be a *monadic extension* of B if the pullback functor

$$p^* : \mathbf{F}(B) \to \mathbf{F}(E)$$

is monadic, or, equivalently, p is an effective descent morphism with respect to the class **F**.

(b) An object (A,f) in $\mathbf{F}(B)$ is said to be a *trivial covering* of B if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

is a pullback, or, equivalently, the morphism $(\eta^B)_{(A,f)} : (A,f) \to H^B I^B(A,f)$ is an isomorphism.

(c) An object (A,f) in $\mathbf{F}(B)$ is said to be *split* over a monadic extension (E,p) of B, if $p^*(A,f)$ is a trivial covering of E.

(d) An object (A,f) in $\mathbf{F}(B)$ is said to be a *covering* of B if it is split over some monadic extension; we then also say that $f : A \to B$ is a covering morphism.

(e) A monadic extension (E,p) is said to be a *normal extension* if it is split over itself.

Remark 1.2. There is a long list of known examples, which we do not recall here. Let us, however, mention that the main ingredient of a Galois structure is of course the adjunction (1.1) that is usually a reflection. In particular the following types of reflections seem to be especially important:

(a) Totally disconnected reflections, where $I(C)$ is the object of connected components of C in a suitable sense. These types of reflections produce classical examples mentioned at the beginning of Introduction.

(b) Reflections of varieties of universal algebras into their subvarieties, or, more generally, reflections of exact categories into their Birkhoff subcategories [JK]. Their covering morphisms are generalized central extensions in the sense of [JK], and in particular central extensions of Ω-groups in the sense of A. S.-T. Lue [L], who also refers to A. Fröhlich's work. A further generalization is developed in [G2].

(c) Torsion-free reflections associated with torsion theories, whose covering morphisms are studied in this paper, continuing [GR].

2. Covering morphisms under presence of a test functor

Definition 2.1. Let $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\Phi)$ be as above. A *test functor* is a finite limit preserving functor $T : \mathbf{C} \to \mathbf{Y}$ from **C** to any category **Y** with finite limits, such that the following conditions on a fibration $f : A \to B$ in **C** are equivalent:

(a) (A,f) is a trivial covering of B;

(b) $T(f) : T(A) \to T(B)$ is an isomorphism in **Y**.

We will fix such a test functor T; the reasons for introducing it are the following obvious facts:

Proposition 2.2. The following conditions on a fibration $f : A \to B$ and a monadic (E,p) of B are equivalent:

(a) (A,f) is split over (E,p);

(b) the pullback projection $T(E\times_B A) \approx T(E)\times_{T(B)}T(A) \to T(E)$ is an isomorphism. □

Proposition 2.3. The following conditions on a monadic extension (E,p) of B are equivalent:

(a) (E,p) is a normal extension;

(b) the pullback projections $T(E\times_B E) \approx T(E)\times_{T(B)}T(E) \to T(E)$ are isomorphisms;

(c) $T(p) : T(E) \to T(B)$ is a monomorphism. □

From now on we will assume that the category **C** is pointed, write 0 for its zero object and its zero morphisms, and write $\ker(f) : \mathrm{Ker}(f) \to A$ for a (the) kernel of a morphism $f : A \to B$ in it. We will also assume that all morphisms into 0 are fibrations. Furthermore, since the functor I must preserve zero, the admissibility condition implies that the functor H is fully faithful, and we will identify the category **X** with its replete H-image in **C**.

Proposition 2.4. The following conditions on an object C in **C** are equivalent:

(a) C is in **X**, i.e. the morphism $\eta_C : C \to HI(C)$ is an isomorphism;

(b) the zero morphism $C \to 0$ is a trivial covering;

(b) $T(C) = 0$. □

Proposition 2.5. For a fibration $f : A \to B$ in **C** the implications (a)⇒(b)⇒(c) hold for:

(a) (A,f) is a normal extension;

(b) (A,f) is a covering;

(c) $\mathrm{Ker}(f)$ is in **X**.

Moreover:

(d) if (A,f) is a monadic extension and every morphism in **Y** with zero kernel is a monomorphism, then conditions (a), (b), and (c) are equivalent to each other;

(e) if there exists a monadic extension (E,p) of B with E in **X**, then condition (c) implies condition (b).

Proof. The implication (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c): When (A,f) is a covering, $p^*(A,f) = (E\times_B A,\mathrm{pr}_1)$ is a trivial covering for some monadic extension (E,p) of B. Since the class of trivial covering morphisms is (obviously) pullback stable, this makes $\mathrm{Ker}(\mathrm{pr}_1 : E\times_B A \to E) \approx \mathrm{Ker}(f) \to 0$ a trivial covering, and we can apply Proposition 2.4.

(d) follows from Proposition 2.3.

(e): We have $T(E\times_B A) \approx T(E)\times_{T(B)}T(A) = 0\times_{T(B)}T(A) = \mathrm{Ker}(T(f)) \approx T(\mathrm{Ker}(f))$, which tell us that $E\times_B A$ is in **X** if and only if so is $\mathrm{Ker}(f)$. But having E and $E\times_B A$ in **X** implies that (A,f) is split over (E,p) and therefore is a covering.□

3. Galois structures of torsion theories

In this section we construct a finitely complete admissible Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ equipped with a test functor T as follows:

• **C** is a homological category in the sense of [BB];

• $(I,H,\eta,\varepsilon) : \mathbf{C} \to \mathbf{X}$ is the torsion-free reflection of a torsion theory $(\mathbf{Y},\mathbf{X})$ on **C** in the sense of [BG] (which generalizes the classical, i.e. abelian, case of S. E. Dickson [D]; we do not consider here the most general context of [JT]);

• **F** and **Φ** are the classes of all morphisms in **C** and **X** respectively;

• $T : \mathbf{C} \to \mathbf{Y}$ is the torsion coreflection of the torsion theory $(\mathbf{Y},\mathbf{X})$ above.

We need conditions (a) and (b) of Definition 2.1 to be equivalent to each other. For, given a morphism $f : A \to B$ in **C**, consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(A) & \xrightarrow{\iota_A} & A & \xrightarrow{\eta_A} & I(A) & \longrightarrow & 0 \\
 & & {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I(f)} & & \\
0 & \longrightarrow & T(B) & \xrightarrow[\iota_B]{} & B & \xrightarrow[\eta_B]{} & I(B) & \longrightarrow & 0,
\end{array}$$

where ι is the counit of the torsion coreflection. We need to know that the right-hand square is a pullback if and only if the first vertical arrow is an isomorphism. However, since the rows of this diagram are short exact sequences and the category **C** is homological, this follows from Bourn protomodularity (see [BB]).

For this Galois structure Γ associated with the torsion theory (**Y**,**X**), the main result of Section 2 becomes:

Theorem 3.1. For a morphism $f : A \rightarrow B$ in **C** the implications (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) hold for:

(a) (A,f) is a monadic extension, and f induces a monomorphism $T(f) : T(A) \rightarrow T(B)$ in **Y** from the torsion coreflection of A to the torsion coreflection of B;

(b) (A,f) is a normal extension;

(c) (A,f) is a covering;

(d) Ker(f) is torsion free.

Moreover:

(e) if (A,f) is a monadic extension and every morphism in **Y** with zero kernel is a monomorphism, then conditions (a), (b), (c) and (d) are equivalent to each other;

(f) if there exists a monadic extension (E,p) of B with E in **X**, then condition (d) implies condition (c). □

Such a theorem obviously suggests to:

• Find non-trivial examples where 3.1(e) and 3.1(f) can be applied; here "non-trivial" has a specific meaning: not every covering should be trivial (in the sense of Definition 1.1(b)).

• Show that conditions 3.1(a), 3.1(b), and 3.1(c) in general are not equivalent to each other. More precisely, find an example where (A,f) satisfies 3.1(d) but not 3.1(c), and another example where it is monadic and satisfies 3.1(c) but not 3.1(b).

This will be done in Sections 4 and 5 respectively.

Remark 3.2. Theorem 3.1 certainly applies to protolocalisations of homological categories in the sense of [BCGS]. However, in that case the functor I preserves all pullbacks along regular epimorphisms, which makes every covering trivial.

4. Two "non-trivial" examples where conditions 3.1(e) and 3.1(f) do apply

Example 4.1. The additional assumption on **Y** made in 3.1(e) obviously holds in the following two cases:

(a) When **Y** is hereditary in **C** with respect to normal monomorphisms, i.e. when every normal monomorphism $m : C \to Y$ in **C** with Y in **Y** must have C in **Y**. It follows that Theorem 3.1 includes the main result in [GR], which was obtained in the more restrictive context of quasi-hereditary torsion theories.

(b) When **C** is (regular and) additive – since this forces **Y** to be additive. In particular **C** could be abelian as in [CJKP].

For both of these cases "the simplest" example of a non-trivial covering morphism (which is even a normal extension) $f : A \to B$ has

• **C** = the category of abelian groups;

• **X** = the category of torsion free abelian groups (in the usual sense);

• f = any epimorphism from the additive group of integers to a non-zero cyclic group.

For this ordinary torsion theory of abelian groups 3.1(f) also applies since every abelian group is a quotient of a torsion-free abelian group. However, it does not apply well to e.g. the dual torsion theory (since only torsion abelian groups are subgroups of torsion abelian groups; see [CJKP] for some related remarks).

Example 4.2. Let $\mathcal{T}$ be a semi-abelian algebraic theory, i.e. an algebraic theory whose models form a semi-abelian category, or, equivalently, a pointed *BIT speciale* variety in the sense of A. Ursini [U] (see also [BouJ], [JMT2], and [JMU] for the clarification of the relationship between the categorical and universal-algebraic approaches). The models of $\mathcal{T}$ are universal algebras of a fixed type admitting, among others, a constant 0, n binary terms $s_1, ..., s_n$, and an $(n + 1)$-ary term t, satisfying the identities

$$t(s_1(x,y),...,s_n(x,y),y) = x, \quad s_1(x,x) = ... = s_n(x,x) = 0, \quad u = 0$$

for all 0-ary terms u. Note that these terms also satisfy the implication

$$s_1(x,y) = ... = s_n(x,y) = 0 \Rightarrow x = y. \tag{4.1}$$

We take

• **C** = the category of topological $\mathcal{T}$-algebras (=models of $\mathcal{T}$ in the category of topological spaces), which is known from F. Borceux and M. M. Clementino [BC] to be a homological category;

• **X** = the category of totally disconnected topological $\mathcal{T}$-algebras, hence obtaining the torsion theory (**Y**,**X**), whose torsion objects are connected topological $\mathcal{T}$-algebras.

We claim that every morphism $f : A \to B$ in the category **Y** of connected topological $\mathcal{T}$-algebras that has the trivial kernel in **Y** is a monomorphism in **Y**. Indeed:

Let g, $h : Y \to A$ be morphisms in **Y** with $fg = fh$, and $k_1, ..., k_n : Y \to A$ be maps (which are not necessarily $\mathcal{T}$-algebra homomorphisms) defined by

$$k_i(y) = s_i(g(y),h(y)) \quad (i = 1, ..., n).$$

Our next step requires to compare the kernel of f in **Y** with the kernel of f in **C**, and we will denote these kernels by $\mathrm{Ker}_{\mathbf{Y}}(f)$ and $\mathrm{Ker}_{\mathbf{C}}(f)$ respectively. We observe:

• $fg = fh$ easily implies that $fk_i = 0$ for all $i = 1, ..., n$.

• Since $\mathrm{Ker}_{\mathbf{C}}(f) = \{a \in A \mid f(a) = 0\}$ and $fk_i = 0$ for all $i = 1, ..., n$, the images of Y under all k_i's are in $\mathrm{Ker}_{\mathbf{C}}(f)$.

• Since $\mathrm{Ker}_{\mathbf{Y}}(f)$ is nothing but the connected component of 0 in $\mathrm{Ker}_{\mathbf{C}}(f)$, and since Y is connected, the previous observation implies that the images of Y under all k_i's are in $\mathrm{Ker}_{\mathbf{Y}}(f)$.

• Therefore $\mathrm{Ker}_{\mathbf{Y}}(f) = 0$ implies $k_i = 0$ for all $i = 1, ..., n$, which itself implies $g = h$ by the implication (4.1).

That is $\mathrm{Ker}_{\mathbf{Y}}(f) = 0$ implies that f is a monomorphism.

There are many non-trivial coverings, e.g. the canonical map R → R/Z in the case when $\mathcal{T}$ is the theory of groups, where R is the topological (additive) group of real numbers and Z is the group of integers. This is, again, a normal extension, and, topologically it is nothing but the classical universal covering of the 1-dimensional sphere S^1 = R/Z of course.

Remark 4.3. Concerning 3.1(f): We do not know if, in the situation 4.2, for every object B in **C**, there exists a monadic extension (E,p) of B with E in **X**. However, as mentioned in [GR] with a reference to A. Arkhangel'skiĭ [A], this is the case when $\mathcal{T}$ is the theory of groups.

5. Counter-examples for (d)⇒(c)⇒(b) in Theorem 3.1

Example 5.1. Let $\mathbf{B}_n$ be the Burnside variety (of groups) of exponent n, i.e. the variety of all groups G with $x^n = 1$ for all x in G, and C_n the cyclic group of order n. We construct the data described in Section 3 by taking $\mathbf{C} = \mathbf{B}_6$ and $\mathbf{X} = \mathbf{B}_3$, and take f to be the unique epimorphism from the symmetric group S_3 to C_2. Then $\mathrm{Ker}(f) = C_3$ is torsion free, but (S_3,f) is not a covering. Indeed:

Consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{\;pr_2\;} & S_3 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ C_6 & \xrightarrow[\;p\;]{} & C_2 \end{array}$$

in which p is the unique epimorphism $C_6 \to C_2$. An easy calculation shows that the projection $P \to C_6$ induces an isomorphism $I(P) \approx I(C_6)$ of the torsion-free reflections; since that projection itself is not an isomorphism, it follows that (P,pr_1) is not a trivial covering. On the other hand, since the category $\mathbf{C} = \mathbf{B}_6$ is exact, and since C_6 is a projective object in it with respect to regular epimorphisms, if (S_3,f) were a covering then it would be split over (C_6,p), i.e. (P,pr_1) would be a trivial covering.

Example 5.2. Now we construct the data described in Section 3 by taking:

- $\mathbf{C}$ = the category of groups;

- $\mathbf{X}$ = the category of torsion-free groups in the usual sense, which will make $\mathbf{Y}$ to be the category of all groups generated by their elements of finite order.

Let us also take $B = C_2$ (=the cyclic group of order 2), $A = B \ltimes Z$ = the infinite dihedral group, and $f : A \to B$ to be the semidirect product projection. In particular f is a split epimorphism and (A,f) is a monadic extension. Consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{pr_2} & A \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow[p]{} & B \end{array}$$

in which p is the unique epimorphism from the additive group of integers to B. Since Z and P are (obviously) torsion free, and (Z,p) is a monadic extension (since p is a regular epimorphism in an exact category), we conclude that (A,f) is a covering of B. On the other hand, since f is a split epimorphism, if (A,f) were a normal extension it would be a trivial covering. And (A,f) is not a trivial covering since A and B are in **Y**, while f is not an isomorphism.

That is, (A,f) is a covering that is a monadic extension but not a normal extension.

6. Remarks on central extensions and locally semisimple coverings

In this section we compare three contexts, which we will call TT, CE, and LSC for short. They are:

- TT: The context of a torsion theory described in Section 3.

- CE stands for "central extension"; it is the context used in [JK], where the ground Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ has **C** an exact category, **X** a Birkhoff subcategory in **C**, and **F** and **Φ** are the classes of regular epimorphisms in **C** and **X** respectively. When **C** is pointed, we could try to define a test functor $T : \mathbf{C} \to \mathbf{Y}$, where **Y** is a suitable subcategory in **C**, by $T(C) = \mathrm{Ker}(\eta_C)$. Moreover, such a functor was actually used in [JK] (called R there) in results similar to ours. However, in general this functor will not preserve finite limits.

• LSC stands for "locally semisimple covering"; it is a context used in [JMT1], where still having (exact) **C** and a full subcategory **X** in it, we do not require the existence of the reflection $I : \mathbf{C} \to \mathbf{X}$. One then replaces trivial coverings with morphisms in **X** and modifies definitions of Section 1 accordingly. So, a locally semisimple covering is a morphism in **C** that is "in **X** up to effective descent" rather than a "trivial covering up to effective descent".

Let us begin our comparisons with the property

$((A,f)$ is a monadic extension and a covering$)$ $\Rightarrow$ $((A,f)$ is a normal extension$)$, (6.1)

which we obtained in TT under any of the following conditions:

Every morphism in **Y** with zero kernel is a monomorphism, (6.2)

There exists a monadic extension (E,p) of B with E in **X**. (6.3)

The implication (6.1) does not make much sense in LSC (unless A is in **X**), but it holds in CE under additional conditions that are (when **C** is exact) much weaker (see [JK]) than what we require in TT. And (6.2) holds in CE as soon as the category **C** has the similar property. Still, in order to prove it in CE, neither the arguments we used for 3.1(e) nor the arguments we used for 3.1(f) can be applied. The reason is that neither 3.1(a)⇒(b) nor (6.3) can be used.

Now let us consider the property

$(\mathrm{Ker}(f)$ is in $\mathbf{X}) \Rightarrow ((A,f)$ is a covering$)$. (6.4)

It "almost never" holds in CE: for instance it does not hold for the ordinary central extensions of groups, which is a very basic fact in group theory (not every group extension with an abelian kernel is central!); an example where it does hold is given by the Birkhoff subcategory Dis(**C**) of discrete

equivalence relations in the category Grpd(**C**) of internal groupoids in a semi-abelian category **C** studied in [G1] and [EG].
But it holds in many special cases of LSC for essentially the same reasons as for 3.1(f). Moreover, 3.1(f) is a consequence of Proposition 2.3 in [JMT1] (in the case of exact **C**, although as mentioned in [JMT1], exactness is not essential there). Indeed, it is easy to see that **X** in TT is a semisimple class in **C** (when **C** is exact) in the sense of [JMT1] satisfying Condition 2.2 of [JMT1], as required in Proposition 2.3 there.

Finally, note that our examples, especially 5.1 and 5.2, are, in a sense, suggested by these comparisons; in fact Example 5.1 can be used also in the context CE and Example 5.2 can be used also in the context LSC (since (6.3) holds there).

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INFINITARY LINEAR COMBINATIONS IN REDUCED COTORSION MODULES

by Reinhard BÖRGER and Ralf KEMPER

Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract

We investigate sets with infinitary linear combinations subject to the usual axioms with coefficients in a suitable ring, e.g. a complete valuation ring. They are Eilenberg-Moore algebras for a monad of countable arity. Moreover, they are always modules; surprisingly infinitary linear combinations yield a *property*. This is quite different from real or the complex case studied by Pumplün and Röhrl.

These modules were called cotorsion modules and defined by a cohomological property by Matlis. They form a reflective subcategory; the reflection also has a cohomological description. This yields some insight, particularly if the first Ulm functor does not vanish.

Nous étudions des ensembles avec combinaisions linéaires infinies, qui satisfont aux axiomes ordinaires, ayant des coefficients dans un anneau avec certaines propriétés, p.ex. un anneau complet d'évaluation. Ici, il s'agit d'algèbres d'Eilenberg-Moore pour une monade d'arité dénombrable. En plus elles sont toujours des modules; de manière inattendue combinaisions linéaires infinies impliquent une *propriété*. C'est tout à fait différent du cas réel ou complexe considéré par Pumplün et Röhrl.

Ces modules de cotorsion étaient définit par une propriété cohomologique par Matlis. Ils constituent une sous-catégorie réflexive; la réflexion a une description cohomologique. Cela nous ouvre des perspectives, en particulier si le premier foncteur d'Ulm ne disparaît pas.

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Key words and phrases: (Krull-)valuation, valuation ring, reduced Matlis-cotorsion module, divisible element, Ulm functor, torsion element, torsion-free module, K–Banach space, monad, Eilenberg–Moore category.

MOS classification: 12J10, 13C05, 13C12, 18E10.

1 Introduction

Pumplün and Röhrl [10] introduced the category **TC** of *totally convex spaces* as the *Eilenberg-Moore category* of the monad induced by the adjunction between the unit-ball functor $\mathbf{Ban}_1 \longrightarrow \mathbf{Set}$ and its left adjoint, where $\mathbf{Ban}_1$ is the category of Banach spaces and linear operators of norm ≤ 1 over the field $\mathbb{R}$ or $\mathbb{C}$. For Banach spaces over a complete field with a *Krull-valuation* subject to some mild conditions the monad and its Eilenberg-Moore-algebras can be formed analogously and form a *locally countably presentable category*, but they look quite different. As opposed to the real and to the complex situation, the Eilenberg-Moore-algebras admit an *addition* subject to the usual rules; this leads to an *additive* and even *abelian* category. The algebras are modules over the valuation ring, and finitary linear combinations are formed as in the module. The existence of infinitary linear combinations excludes the existence of non-trivial divisible submodules; following some authors we call a module with this property *reduced*. The module carries a canonical topology, which turns out to be bounded, and every infinitary linear combination is the limit of the finitary sub-combinations in this topology. If the topology is Hausdorff, this limit is unique; this happens if and only if the module is *division-free*, i.e. the first *Ulm functor* vanishes. But there also exist division elements in Eilenberg-Moore-algebras; surprisingly, then the infinitary linear combinations are still determined by the finitary ones. The Eilenberg-Moore-category is even a *full* subcategory of the category of all modules. Its objects were already investigated by Matlis [9] in a different context and called *cotorsion modules*; according to new terminology we call them *reduced* Matlis-cotorsion modules. It is well-known that Matlis-cotorsion is equivalent to completeness in the division-free situation, but surprisingly, our results remain valid if there exist non-trivial division elements. Then infinitary linear combinations are still uniquely determined, though the limit of the corresponding finitary linear combinations is not unique. Infinitary linear combinations can also be characterized as unique solutions of systems of linear equations. Though always some Ulm functor vanishes, the chain of Ulm functors can be arbitrarily long. This easily implies that the category of reduced Matlis-cotorsion modules has no cogenerator.

2 Eilenberg-Moore-algebras

In this section we consider a field K with a *Krull–valuation*, i.e. a surjective map $v : K \longrightarrow \overline{\Gamma}$, where $\overline{\Gamma} := \Gamma \cup \{\infty\}$, Γ a *totally ordered* (additively written) *abelian group* and ∞ is an additional largest element, subject to the following axioms:

(V1) $v(\alpha) = \infty$ if and only if $\alpha = 0$.

(V2) $v(\alpha\beta) = v(\alpha) + v(\beta)$ for all $\alpha, \beta \in K$.

(V3) $v(\alpha + \beta) \geq \min(v(\alpha), v(\beta))$ for all $\alpha, \beta \in K$.

Surjectivity of v can always be achieved by codomain restriction. We want to include the case that the value group Γ is not archimedean; the valuation is non-archimedean anyway. The totally ordered group Γ is archimedean if and only if it can be embedded into $\mathbb{R}$; the most famous examples are p-adic valuations. In order to avoid some trivial cases, we assume that Γ contains a countable unbounded subset; this guarantees the existence of convergent subsequences which are not eventually constant. Since the single-element group is obviously bounded, Γ has at least two elements; this makes the canonical topology non-discrete. The above conditions are always satisfied for non-trivial real-valued valuations. Moreover, we assume that K is *complete*, i.e. sequence $(\alpha_n)_{n\in\mathbb{N}}$ in K converges if $(\alpha_n - \alpha_{n+1})_{n\in\mathbb{N}}$ converges to 0; we always assume $0 \in \mathbb{N}$. Here a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges to $\alpha \in K$ if $v(\alpha_n - \alpha) \to \infty$ for

$n \to \infty$. The *valuation ring* $R := \{\alpha \in K \mid v(\alpha) \geq 0\}$ is a local ring with *maximal ideal* $\{\alpha \in K \mid v(\alpha) > 0\}$.

Let us now define (K, v)-Banach spaces; in the case of a real-valued field they were defined by A.F. Monna and studied by van Rooij [11]. This leads to an easy generalization to this case; assuming $||\alpha|| := e^{-v(\alpha)}$ for $\alpha \neq 0$ and $||0|| := 0$. Then the (K, v)-Banach spaces form a complete and cocomplete category; the construction of limits and colimits sometimes requires suprema and infima which exist by completeness of $\mathbb{R}$. For non-archimedean Γ this is not possible, because then Γ is not complete and cannot even be embedded into a complete totally ordered group. Therefore, we define the Banach structure on a K-vector space E by a binary relation $\dashv$ on $E \times \Gamma$; here $x \dashv g$ should be thought of as $u(x) \geq g$ for some (valuation) map $u : E \to \overline{\Gamma}$; for a real-valued v we can define (K, v)-Banach spaces by v in this way. Another problem occurs: If Γ has no least positive element, then 0 is the infimum of all positive elements, and every element of Γ is the supremum of all strictly smaller elements and the infimum of all strictly larger elements. If Γ has a smallest positive element, this is not the case. This requires a distinction of two cases; we come to the following definition:

A (K, v)-*Banach space* is a K-vector space E together with a binary relation $\dashv$ on $E \times \Gamma$ with:

(KB0) For every $x \in E$ there exists a $g \in \Gamma$ with $x \dashv g$. $0 \dashv g$ holds for all $g \in \Gamma$, but for every $x \in E \setminus \{0\}$ there exists a $g \in \Gamma$ with $x \not\dashv g$.

(KB1) $x \dashv g'$ whenever $x \dashv g$ and $g \geq g'$.

(KB2) $\alpha x \dashv v(\alpha) + g$ whenever $x \dashv g$, $\alpha \in K \setminus \{0\}$.

(KB3) $x + y \dashv g$ if $x \dashv g$ and $y \dashv g$.

(KB4) If $(x_n)_{n \in \mathbb{N}}$ is a sequence in E such that for every $g \in \Gamma$ there exists an $n_0 \in \mathbb{N}$ with $x_n - x_{n+1} \dashv g$ for all $n \geq n_0$, then there exists an $x \in V$ such that for every $g \in \Gamma$ there exists an $n_1 \in \mathbb{N}$ with $x_n - x \dashv g$ for all $n \geq n_1$.

If Γ has a least positive element (but only in this case) we also assume:

(KB5) $x \dashv 0$ whenever $x \dashv g$ for all positive $g \in \Gamma$.

A *morphism* $f : E_0 \to E_1$ of (K, v)-Banach spaces is a K-linear map $E_0 \to E_1$ such that $x \dashv g$ implies $f(x) \dashv g$.

Observe that (KB4) is a *completeness* condition; it means that every Cauchy sequence in the *canonical topology* converges; the canonical topology has the sets $U_g := \{x; x \dashv g\}$ as a basis. The Banach structure is already given by the *unit ball* $\bigcirc E := \{x \in E | x \dashv 0\}$ of E; observe $x \dashv v(\alpha)$ if and only if $\alpha^{-1}x \in \bigcirc E$ for $\alpha \neq 0$. $\bigcirc E$ is no longer a K-vector space, but still an R-module. Observe that the base-field K is always a (K, v)-Banach space in the canonical way with $\bigcirc K = R$. By the existence of a countable unbounded set in Γ this topology is always first-countable and hence sequential, i.e. every sequentially closed set is closed.

The set-valued functor $\bigcirc$ on the category of (K, v)-Banach spaces has a left adjoint and induces a monad on the category of sets. This left adjoint maps every set X to the K-vector space $\ell_1(X)$ of all families $(\xi_x)_{x\in X} \in K^X$ such that for each $g \in \Gamma$ there are only finitely many $x \in X$ with $v(\xi_x) < g$; we define $(\xi_x)_{x\in X} \dashv g :\Leftrightarrow \forall x \in X\ v(\xi_x) \geq g$ for $(\xi_x)_{x\in X} \in \ell_1(X)$ and $g \in \Gamma$. An Eilenberg-Moore-algebra is a non-empty set M together with maps $M^{\mathbb{N}} \to M$ for $(\alpha_n)_{n\in\mathbb{N}} \in \Omega$, which we shall write as $x_\bullet := (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} \alpha_n x_n$, where $\Omega := \ell_1(\mathbb{N}) = \{\alpha_\bullet = (\alpha_n)_{n\in\mathbb{N}} | \ \forall n \in \mathbb{N}\ \alpha_n \in R,\ v(\alpha_n) \to \infty \text{ for } n \to \infty\}$. For $\alpha_\bullet = (\alpha_n)_{n\in\mathbb{N}} \in \Omega$ we put $\alpha_\bullet^\odot := (\alpha_{n+1})_{n\in\mathbb{N}} \in \Omega$; likewise we define $x_\bullet^\odot := (x_{n+1})_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ for $x_\bullet := (x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$. Moreover, for $\beta \in K$ and $\alpha_\bullet \in \Omega$ we define $\beta\alpha_\bullet := (\beta\alpha_n)_{n\in\mathbb{N}}$; it even belongs to Ω if $\beta\alpha_n \in R = \bigcirc K$ for all $n \in \mathbb{N}$; this is always satisfied for $\beta \in R$. Since every $\alpha_\bullet \in \Omega$ converges to 0 by hypothesis, for each $\beta \in R\setminus\{0\}$ there are only finitely many $n \in \mathbb{N}$ with $\beta^{-1}\alpha_n \notin R$; therefore iterated application of $^\odot$ to $\beta^{-1}\alpha_\bullet$ finally leads to an element of Ω. There is always a distinguished element $0 = \sum_{n=0}^{\infty} 0x_n \in M$, which does not depend on the $x_n \in M$. The Eilenberg-Moore-algebras for the adjunction given by the above unit ball functor $\bigcirc$ and its left adjoint ℓ_1 can be characterized in our situation in the same way as Pumplün and Röhrl did in the real and in the complex situation (cf. [10]).

Theorem 2.1 *The Eilenberg-Moore-category of the monad induced by the set-valued functor $\bigcirc$ is the category of non-empty sets M together with maps $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} \alpha_n x_n$ for $(\alpha_n)_{n\in\mathbb{N}} \in \Omega$ subject to the follow-*

ing axioms:

(LC1) $\sum_{n=0}^{\infty} \delta_{n,m} x_n = x_m$ *for all* $m \in \mathbb{N}$*, where* δ *is the Kronecker symbol with values in* R*, i.e.* $\delta_{n,n} = 1$ *and* $\delta_{n,k} = 0$ *for* $n \neq k$*.*

(LC2) $\sum_{n=0}^{\infty} \alpha_n (\sum_{m=0}^{\infty} \beta_{n,m} x_m) = \sum_{m=0}^{\infty} (\sum_{n=0}^{\infty} \alpha_n \beta_{n,m}) x_m$ *for all* $\alpha_\bullet \in \Omega$*, and* $\beta_{n,\bullet} \in \Omega$ *for all* $n \in \mathbb{N}$*, where* $x_m \in M$ *for all* $m \in \mathbb{N}$*.* □

The proof is completely analogous to the real and in the complex situation; nevertheless, the algebras look quite different. As in the real and in the complex case, the monad has rank $\aleph_1$; all operations are countable; therefore the category is locally countably presentable (cf. [6]). Moreover Ω contains all sequences in R with only finitely many non-zero entries; this defines *finitary linear operations* in every algebra M; so M becomes an *R-module*. In this way the Eilenberg-Moore-category is *additive*; it even turns out to be *abelian*; it is also a symmetric monoidal closed category, even an autonomous category in the sense of Linton [8]. The tensor product in this category can be constructed by applying the reflection to the usual tensor product of modules. The set-theoretical image of every morphism is a subalgebra, in particular a submodule. This submodule can always be divided out; so we see that all epimorphisms are surjective. All this is different from the real and the complex case because $1 + 1 = 2 > 1$ holds there; thus the category is no longer additive. These algebras were studied in more detail in the *habilitationsschrift* of the second author [7].

We can also define (K, v)*-normed vector spaces* as above by omitting the completeness condition (KB4); we do not need the completeness of K. Instead of Ω we get the set of all sequences in R with only finitely many non-zero entries. Now the operations can be viewed as *finitary* linear combinations; in particular, we have a binary addition and a multiplication with an element of R on each Eilenberg-Moore-algebra M. Then we easily see that M is an R-module under these operations, and arbitrary linear combinations are just as in this module. Conversely, in every R-module the finitary linear combinations satisfy (LC1) and (LC2). Therefore the Eilenberg-Moore-category is just the category of R-modules in this situation.

If $\sigma : \Gamma \to \Gamma'$ is a surjective order-preserving group homomorphism, and if we still have $\Gamma' \neq \{0\}$, then the valuation $v' := \sigma \circ v$ also satisfies the above conditions. The kernel of σ is then bounded, but the valuation ring R' of v' is strictly larger than R unless σ is bijective, because there are elements $g < 0$ in Γ with $\sigma(g) = 0$. Every (K, v)-Banach space E is a (K, v')-Banach space in the obvious way; it carries the same canonical topology, but as a (K, v')-Banach space it has a larger unit ball. In particular, R is the unit ball of K over R and admits infinitary linear combinations with (LC1) and (LC2) over R, but not over R'.

3 Finitary and infinitary linear combinations

In the remainder of this paper we shall consider a more general situation. The ring R need not be a valuation ring. But we assume that R is an integral domain and $K \neq R$ is its quotient field. More generally, every R-module M carries a *canonical topology*, whose basis are all sets αM with $\alpha \in R \setminus \{0\}$. For a valuation v it coincides with our previous definition. Since R is not a field, the canonical topology on R is Hausdorff. This happens for valuations because we consider surjective Krull-valuations rather than (possibly non-surjective) valuations into $\mathbb{R}$; for the latter approach, a trivial valuation would lead to the discrete topology. Moreover, we assume R to be *first countable* in the canonical topology; this is equivalent to saying that R is *powerful* in the sense of Matlis [9], i.e. K is countably generated as an R-module; then K has homological dimension 1, as we shall see: K is a union of an increasing chain of countably many submodules $\gamma_n^{-1}R$ with $\gamma_n \in R \setminus \{0\}$ for all $n \in \mathbb{N}$; we also can assume $\gamma_0 = 1$. This can always be achieved by choosing γ_n as the product of the denominators of the first n generators. Each one of the modules $\gamma_n^{-1}R$ is isomorphic to R. If R is a valuation ring for a valuation $v : K \to \overline{\Gamma}$ and if the set of integer multiples of $v(\gamma)$ is unbounded in R for some $\gamma \in R \setminus \{0\}$, then we can choose $\gamma_n := \gamma^n$ for all $n \in \mathbb{N}$; this yields $\gamma_n^{-1}\gamma_{n+1} = \gamma$ for each $n \in \mathbb{N}$. In the non-powerful case the situation may be much more complicated.

In particular, we have a short exact sequence $0 \to R^{(\mathbb{N})} \to R^{(\mathbb{N})} \to$

$K \to 0$, where the morphism $R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ maps the n-th unit vector e_n to $e_n - (\gamma_n^{-1}\gamma_{n+1})e_{n+1}$; the other non-trivial map is $R^{(\mathbb{N})} \to K$, $e_n \mapsto \gamma_n^{-1}$. M is also *sequential* in the canonical topology; i.e. every sequentially closed subset of M is closed. Finally we assume R to be *(sequentially) complete* in the canonical topology; this means that every sum in R converges in the canonical topology provided its members converges to 0. This is more general as the case of a valuation ring, but it cannot always be achieved by completion; e.g. the completion of the powerful integral domain $\mathbb{Z}$ has zero-divisors; it is the product of all rings of p-adic integers for all primes p.

For valuations v, v' and a surjection σ as above, every R'-module is an R-module by restriction of the operations, and one easily sees that the canonical topologies coincide for both valuations.

The first *Ulm functor* $U = U^1$ is defined by $UM := \bigcap_{\alpha \in R\setminus\{0\}} \alpha M$ for every R-module M; this is an R-module again. Moreover, U^0 is the identity functor for R-modules; for each ordinal number κ we define $U^{\kappa+1} := UU^\kappa$, and for a limit ordinal λ we set $U^\lambda M := \bigcap_{\kappa<\lambda} U^\kappa M$. Moreover, we put $U^\infty M := \bigcap_{\kappa \text{ ordinal}} U^\kappa M$. Then M is *divisible* if and only if $UM = M$ holds; M is called *division-free* if and only if $UM = \{0\}$ holds. If M admits infinitary linear combinations with (LC1) and (LC2), then all $U^\kappa M$, in particular $U^\infty M$, are closed under these combinations.

Theorem 3.1 *The division-free R-modules form a full reflective subcategory of the category of all R-modules; the reflection maps M to M/UM.* □

Proposition 3.2 *For an R-module M the following statements are equivalent:*

(i) *M is reduced.*

(ii) $U^\infty M = \{0\}$.

(iii) $\mathrm{Hom}(K, M) = \{0\}$.

Proof. (i) ⇒ (ii): The $U^\kappa M$ form a decreasing sequence of submodules of M; thus it must become constant somewhere. Hence there exists an ordinal κ with $U^\kappa M = U^{\kappa+1}M = UU^\kappa M$; therefore $U^\infty M = U^\kappa M$ is divisible. Now (i) yields $U^\infty M = \{0\}$.

(ii) $\Rightarrow$ (iii): For an R-linear map $f : K \to M$ we obtain $f(K) = f(U^\infty K) \subset U^\infty M = \{0\}$.

(iii) $\Rightarrow$ (i): Let $D \subset M$ be a divisible submodule and assume $x_0 \in D$. We have an increasing representation $K = \bigcup_{n=0}^\infty \gamma_n^{-1}R$ with $\gamma_0 = 1$. Since D is divisible, we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in D with $(\gamma_n^{-1}\gamma_{n+1})x_{n+1} = x_n$ for all $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$ we consider the map $\gamma_n^{-1}R \to M$, $\xi \mapsto \gamma_n\xi x_n$. This map is R-linear, and the map for $n+1$ extends the map for n. So they can be merged to a linear map $f : K \to M$, which is trivial by (iii). This implies $x_0 = f(1) = 0$, proving $D = \{0\}$. □

The statements (i) and (ii) are always equivalent, and they imply (iii), but the proof of (iii) $\Rightarrow$ (i) needs the hypothesis that R be powerful. Modules satisfying (iii) are called h-reduced, e.g. by Matlis [9]. The assumption is necessary in the case of a valuation ring R for a Krull valuation $v : K \to \overline{\Gamma}$. Indeed, if R is not powerful, i.e. if in $\Gamma \neq \{0\}$ every countable subset is bounded; then the homological dimension of K as an R-module is ≥ 2 by VI,3.4 of [3], and from VII, 2.8 of [3] we see that there exists a reduced R-module which is not h-reduced and hence an h-divisible R-module which is not divisible.

The canonical topology of an R-module M has the set of all γM with $\gamma \in R \setminus \{0\}$ as a basis of 0-neighbourhoods; it is Hausdorff if and only if M is *division-free*. The finite sums $\sum_{n=0}^m \alpha_n x_n$ converge to the infinitary linear combination $\sum_{n=0}^\infty \alpha_n x_n$ in the canonical topology for all $\alpha_\bullet \in \Omega$. In the division-free case, the limit is unique, (LC1) and (LC2) are clearly satisfied, and we can split up the infinitary linear combinations into module operations and limits. In general, an infinitary linear combination is *one* operation and the *coefficients* are crucial, not just the *summands*. In particular, it cannot be split into module operations and some unique limits, maybe in a finer topology, as we see in the following

Theorem 3.3 *If an R-module M admits infinitary linear combinations satisfying* (LC1) *and* (LC2)*, then an element x_0 of M belongs to UM if* *and only if it is of the form* $x_0 = \sum_{n=0}^{\infty} \alpha_n y_n$ *with* $\alpha_\bullet \in \Omega$ *and* $\alpha_n y_n = 0$ *for all* $n \in \mathbb{N}$.

Proof. For x_0 of the given form and for $\gamma \in R \setminus \{0\}$ there is an $m \in \mathbb{N}$ such that m applications of $^\odot$ to $\gamma^{-1}\alpha_\bullet$ yield an element of Ω. Then we have

$$\begin{aligned} x_0 = & \sum_{n=0}^{\infty} \alpha_n y_n = \sum_{n=0}^{m-1} \alpha_n y_n + \sum_{n=m}^{\infty} \alpha_n y_n \\ = & \sum_{n=0}^{m-1} 0 + \sum_{n=m}^{\infty} (\gamma^{-1}\alpha_n)(\gamma y_n) = \gamma \sum_{n=m}^{\infty} (\gamma^{-1}\alpha_n) y_n \in \gamma M. \end{aligned}$$

This yields $x_0 \in UM$.

Conversely, we have an increasing representation $K = \bigcup_{n=0}^{\infty} \alpha_n^{-1} R$ with $\alpha_0 = 1$; in particular we have $\alpha_\bullet \in \Omega$. Since $x_0 \in UM$, there are x_n in M with $x_0 = \alpha_n x_n$ for all $n \in \mathbb{N}$. Then for $y_n := x_n - (\alpha_n^{-1}\alpha_{n+1})x_{n+1}$ ($n \in \mathbb{N}$) we obtain

$$\begin{aligned} x_0 = & x_0 + \sum_{n=1}^{\infty} \alpha_n x_n - \sum_{n=1}^{\infty} \alpha_n x_n = \sum_{n=0}^{\infty} \alpha_n x_n - \sum_{n=1}^{\infty} \alpha_n x_n \\ = & \sum_{n=0}^{\infty} \alpha_n x_n - \sum_{n=0}^{\infty} \alpha_{n+1} x_{n+1} = \sum_{n=0}^{\infty} \alpha_n y_n \end{aligned}$$

with $\alpha_n y_n = \alpha_n x_n - \alpha_{n+1} x_{n+1} = x_0 - x_0 = 0$. □

A topological R-module M is called *bounded* if for every 0–neighbourhood $W \subset M$ there is an $\alpha \in R \setminus \{0\}$ with $\alpha M \subset W$. M is called sequentially complete if every Cauchy sequence converges; a sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if for every 0–neighbourhood $W \subset M$ there exists an $n_0 \in \mathbb{N}$ with $x_{n_1} - x_{n_2} \in W$ for all $n_1, n_2 \geq n_0$; since we are in the non-archimedean situation this property of n_0 is equivalent to $x_{n+1} - x_n \in W$ for all $n \geq n_0$. If M is a sequentially complete R-module, then for every $\alpha_\bullet \in \Omega$ the partial sums $\sum_{n=0}^{m} \alpha_n x_n$ converge to $\sum_{n=0}^{\infty} \alpha_n x_n$ for $m \to \infty$.

Proposition 3.4 *Let* M *be a topological* R*–module such that* $(\sum_{n=0}^{m} \alpha_n x_n)_{m\in\mathbb{N}}$ *converges for every* $\alpha_\bullet \in \Omega$, $x_\bullet \in M^{\mathbb{N}}$. *Then* M *is bounded.*

Proof. Assume the contrary. Then there is a 0–neighbourhood $W \subset M$ with $\gamma M \not\subset W$ for all $\gamma \in R \setminus \{0\}$. Since R is powerful, we can find $\alpha_n \in R$, $n \in \mathbb{N}$ with $\bigcup_{n\in\mathbb{N}} \alpha_n^{-1}R = K$; we can even achieve $\alpha_{n+1} \in \alpha_n R$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there is an $x_n \in M$ with $\alpha_n x_n \notin W$; hence $(\alpha_n x_n)_{n\in\mathbb{N}}$ does not converge to 0 in M, therefore $(\sum_{n=0}^{m} \alpha_n x_n)_{m\in\mathbb{N}}$ cannot converge in M, though $\alpha_\bullet \in \Omega$. □

The following observation is crucial for the further discussion: We shall see later that infinitary linear combinations do not guarantee that the canonical topology is Hausdorff, i.e. that the module is *division-free*. But it is still true that the modules are *reduced*, i.e. they may contain non-trivial divisible *elements*, but not divisible *submodules* (which would be closed under these operations by our previous remark).

Theorem 3.5 *Every R-module that admits infinitary linear combinations satisfying* (LC1) *and* (LC2) *is reduced.*

Proof. We use 3.2 (iii) ⇒ (i). For an R-module M let $f : K \to M$ be R-linear. As above, we can represent K as an increasing union $\bigcup_{n=0}^{\infty} \alpha_n^{-1}R$ of copies of R with $\alpha_0 = 1$. Then in M for every $\xi \in K$ we obtain

$$f(\xi) + \sum_{n=1}^{\infty} \alpha_n f(\alpha_n^{-1}\xi) = \sum_{n=0}^{\infty} \alpha_n f(\alpha_n^{-1}\xi) =$$

$$\sum_{n=0}^{\infty} \alpha_n ((\alpha_n^{-1}\alpha_{n+1}) f(\alpha_{n+1}^{-1}\xi)) = \sum_{n=0}^{\infty} \alpha_{n+1} f(\alpha_{n+1}^{-1}\xi) =$$

$$\sum_{n=1}^{\infty} \alpha_n f(\alpha_n^{-1}\xi)\,,$$

hence $f(\xi) = 0$, proving $f = 0$. □

The following observation looks surprising at first glance:

Theorem 3.6 *If M and N admit infinitary linear combinations with* (LC1) *and* (LC2) *and if $f : M \to N$ is an R-module homomorphism, then f preserves these infinitary linear combinations.*

Proof. The set

$$D := \{\sum_{n=0}^{\infty} \alpha_n f(x_n) - f(\sum_{n=0}^{\infty} \alpha_n x_n) | x_n \in M \text{ for } n \in \mathbb{N}, \alpha_\bullet \in \Omega\}$$

is an R-submodule of M; and we have to show $D = \{0\}$; by 3.5 it suffices to show that D is divisible. For an arbitrary

$$y = \sum_{n=0}^{\infty} \alpha_n f(x_n) - f(\sum_{n=0}^{\infty} \alpha_n x_n) \in D,$$

and $m \in \mathbb{N}$ we obtain

$$y = \sum_{n=0}^{m-1} \alpha_n f(x_n) - f(\sum_{n=0}^{m-1} \alpha_n x_n) + \sum_{n=m}^{\infty} \alpha_n f(x_n) - f(\sum_{n=m}^{\infty} \alpha_n x_n) = \sum_{n=m}^{\infty} \alpha_n f(x_n) - f(\sum_{n=m}^{\infty} \alpha_n x_n)$$

because the linear map f preserves finitary linear combinations. For every $\gamma \in R \setminus \{0\}$ there exists an $m \in \mathbb{N}$ with $\gamma^{-1}\alpha_\bullet \star m \in \Omega$. Then we have $z := \sum_{n=m}^{\infty}(\gamma^{-1}\alpha_n) f(x_n) - f(\sum_{n=m}^{\infty}(\gamma^{-1}\alpha_n)x_n) \in D$ and $\gamma z = y$. This proves $y \in UD$, hence D is divisible. □

Corollary 3.7 *On an R-module there is at most one way to introduce infinitary linear combinations with* (LC1) *and* (LC2) *extending the finitary ones.*

Proof. Apply 3.6 to the identity map. □

This is the only case we know, where a full reflective subcategory of an equational locally finitely presentable category is equational (and locally countably presentable), but not locally finitely presentable.

4 Cotorsion Modules

We have seen that infinitary linear combinations are unique in every R-module and are preserved by every R-linear map; so the remaining question is their *existence*. Modules admitting infinitary linear combinations with (LC1) and (LC2) form a full subcategory of the category of all R-modules; we shall see that they coincide with the modules called *cotorsion* modules by Matlis [9], who introduced them. Later it has become common to define this term without reducedness; also the stronger notions of *Enochs-cotorsion* and *Warfield-cotorsion* were introduced; they were studied by Enochs, Fuchs, Harrison, Matlis and Warfield (cf. [2], [3], [5]). M is a *reduced Matlis-cotorsion module* if and only if $\mathrm{Hom}(K, M) = \{0\}$ and $\mathrm{Ext}^1(K, M) = \{0\}$ hold; torsion-free ones are classical examples of splitters (cf. [4]).

The above subcategory is also *reflective*. Once we have characterized it, we can describe the reflection in terms of the long cohomology sequence. But in order to achieve the characterization, we need its existence first and also another property.

Lemma 4.1 *The R-modules admitting infinitary linear combinations with* (LC1) *and* (LC2) *form a reflective subcategory of the category of R-modules. The reflection map is injective if and only if M is reduced.*

Proof. Since the category of modules with infinitary linear combinations with (LC1) and (LC2) is defined by adding more operations and equations to an equationally defined category, the existence of the left adjoint follows from the Adjoint Functor Theorem (cf. [1]).

Obviously, M is reduced by 3.5 if the reflection map $r : M \to M'$ is injective. Conversely, for reduced M consider the pushout of r along the multiplication $M \to M$, $\xi \mapsto \gamma\xi$ for some $\gamma \in R \setminus \{0\}$, i.e. the map $r' : M \to M'' := (M \times M')/N$, $x \mapsto (x, 0) + N$, where $N := \{(\gamma x, -r(x)) | x \in M\} \subset M \times M'$. This yields a module structure on M''; it does not admits infinitary linear combinations with (LC1) and (LC2) for simple categorical reasons. Usually, the definition does not contain the minus sign; but here it does not change the module and it allows to "add the components" in the usual way. The infinitary linear combinations are given by

$$\sum_{n=0}^{\infty} \alpha_n((x_n, y_n) + N) := (\sum_{n=0}^{\infty} \alpha_n r'(x_n, 0)) + (0, \sum_{n=0}^{\infty} \alpha_n y_n) + N$$

and

$$\sum_{n=0}^{\infty} \alpha_n r'(x_n, 0) := (\sum_{n=0}^{m} \alpha_n x_n, \sum_{n=m+1}^{\infty} (\gamma^{-1}\alpha_n) r(x_n)) + N = \sum_{n=0}^{m} \alpha_n r'(x_n, 0) + \gamma \sum_{n=m+1}^{\infty} (\gamma^{-1}\alpha_n) r'(x_n, 0)$$

whenever $\alpha_\bullet \in \Omega$, $x_\bullet \in M^{\mathbb{N}}$, $y_\bullet \in M'^{\mathbb{N}}$,$\gamma \in R \setminus \{0\}$, and $\gamma^{-1}\alpha_n \in R$ for $n > m$; the result does not depend on the choice of m.

Now from the universal property of r we obtain an R-module homomorphism $f : M' \to M''$ with $f \circ r = r'$. This implies that the kernel $\tilde{N}$ of r is contained in the kernel $\gamma\tilde{N}$ of r'. As $\gamma \in R \setminus \{0\}$ is arbitrary, $\tilde{N} \subset M$ is divisible; since M is reduced, this implies $\tilde{N} = \{0\}$. □

Theorem 4.2 *For an R-module M the following statements are equivalent:*

(i) *M admits infinitary linear combinations with* (LC1) *and* (LC2).

(ii) *The following system of linear equations has a unique solution* $(y_{\alpha_\bullet,x_\bullet})_{\alpha_\bullet\in\Omega,x_\bullet\in M^{\mathbb{N}}}$*:*

$$y_{\alpha_\bullet,x_\bullet} = \alpha_0 x_0 + y_{\alpha_\bullet^\odot,x_\bullet^\odot} \text{ for all } \alpha_\bullet \in \Omega,\ x_\bullet \in M^{\mathbb{N}}$$
$$y_{\beta\alpha_\bullet,x_\bullet} = \beta y_{\alpha_\bullet,x_\bullet} \text{ for all } \beta \in R,\ m \in \mathbb{N},\ \alpha_\bullet \in \Omega,\ x_\bullet \in M^{\mathbb{N}}.$$

(iii) *M is a reduced Matlis-cotorsion module, i.e.* $\mathrm{Hom}(K, M) = \{0\}$ *and* $\mathrm{Ext}^1(K, M) = \{0\}$.

Proof. (i) ⇔ (ii): If we have infinitary linear combinations, we can solve the system of equations by $y_{\alpha_\bullet,x_\bullet} := \sum_{n=0}^{\infty} \alpha_n x_n$ for all $\alpha_\bullet \in \Omega$, $m \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$, $\beta \in R$. The solution is unique, because the corresponding homogenous system has only the zero solution. Indeed, if we have a solution for $x_n = 0$ for all $n \in \mathbb{N}$, then the y_n form a divisible submodule of M, which must vanish by (i) and 3.5. Conversely, if we have a solution of the system of equations, we define $\sum_{n=0}^{\infty} \alpha_n x_n := y_{\alpha_\bullet,x_\bullet}$ for $\alpha_\bullet \in \Omega$, $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$. Using the uniqueness, we get (LC1) and (LC2).

(i) $\Rightarrow$ (iii): Consider the *torsion-free cover* F of M; it can be constructed as the module of all homomorphisms f from K into the injective hull J of M with $f(1) \in M$. Then F is torsion-free, and we have the canonical projection $F \to M$; according to the above construction it is the linear map $f \mapsto f(1)$. Its kernel N consists of all $f : K \to J$ with $f(1) = 0$. Moreover, N is torsion-free as a submodule of F.

Now assume that M admits infinitary linear combinations with (LC1) and (LC2); from 3.5 we see that M is reduced. So we still have to show $\mathrm{Ext}^1(K, M) = \{0\}$. For each fixed $\xi \in K$ there is a $\gamma \in R \setminus \{0\}$ with $\gamma\xi \in R$. Then for every $g \in F$ we obtain $\gamma g(\xi) = g(\gamma\xi) = (\gamma\xi)g(1) \in M$. So for ξ and γ as above we can define the map $\sum_{n=0}^{\infty} \alpha_n f_n : K \to J$ in F by $(\sum_{n=0}^{\infty} \alpha_n f_n)(\xi) := \sum_{n=0}^{\infty} \alpha_n(\gamma f_n(\gamma^{-1}\xi))$. This definition does not depend on the choice of γ.

This defines infinitary linear combinations with (LC1) and (LC2) on the torsion-free R-module F. So F is complete in the canonical topology, and from Matlis [9] we obtain $\mathrm{Ext}^1(K, F) = \{0\}$. Since K has projective dimension 1, we have $\mathrm{Ext}^2(K, N) = \{0\}$.

Now we apply the left-exact functor $\mathrm{Hom}(K, -)$ to the short exact sequence

$$0 \to N \to F \to M \to 0.$$

Then the long cohomology sequence contains the part $\mathrm{Ext}^1(K, F) \to \mathrm{Ext}^1(K, M) \to \mathrm{Ext}^2(K, N)$. Since the first and the last module vanish, we can conclude $\mathrm{Ext}^1(K, M) = \{0\}$.

(iii) $\Rightarrow$ (i): Assume $\mathrm{Hom}(K, M) = \{0\}$ and $\mathrm{Ext}^1(K, M) = \{0\}$ and let $r : M \to M'$ be the reflection map from 4.1. Then M is reduced by 3.2, hence r is injective, and we have a short exact sequence

$$0 \to M \to M' \to M'/M.$$

Since M' admits infinitary linear combinations with (LC1) and (LC2), it is also reduced, i.e. $\mathrm{Hom}(K, M') = \{0\}$, and from the long cohomology sequence we see that $\mathrm{Hom}(K, M'/M) = \{0\}$, i.e. M'/M is reduced. But M'/M is also divisible: For all $z \in M'$ and all $\gamma \in R \setminus \{0\}$ we have to show $z + r(M) \in \gamma(M'/r(M)) = \gamma M' + r(M)$. By a routine argument, z is of the form $z = \sum_{n=0}^{\infty} \alpha_n r(x_n)$ for some $\alpha_\bullet \in \Omega$ and some $x_\bullet \in M^{\mathbb{N}}$, because all such elements form a submodule of M' containing $r(M)$ closed under infinitary linear combinations. Then there is an $m \in \mathbb{N}$ with $\gamma^{-1}\alpha_\bullet \star m \in \Omega$, and then we get $z = \sum_{n=0}^{\infty} \alpha_n r(x_n) = \sum_{n=0}^{m-1} \alpha_n r(x_n) + \sum_{n=m}^{\infty} \alpha_n r(x_n) = r(\sum_{n=0}^{m-1} \alpha_n x_n) + \gamma \sum_{n=m}^{\infty} (\gamma^{-1}\alpha_n) r(x_n) \in r(M) + \gamma M'$.

So M'/M is both divisible and reduced, thus we have $M'/M = \{0\}$. This means that r is bijective; since M' admits infinitary linear combinations with (LC1) and (LC2), M also does. □

For an individual $\alpha'_\bullet \in \Omega$ and $x'_\bullet \in M^{\mathbb{N}}$ the linear combination $\sum_{n=0}^{\infty} \alpha'_n x'_n$ can be uniquely characterized by a countable system of linear equations. In order to determine $y_{\alpha_\bullet, x_\bullet}$ it suffices to have the equations in (ii) for countably many cases.

For an increasing representation $K = \bigcup_{n=0}^{\infty} \alpha_n^{-1} R$ we need only the countably many cases where β is α_m for some $m \in \mathbb{N}$ and $\beta\alpha_\bullet$ is obtained from $\alpha'_\bullet$ by finitely many applications of $^\odot$ and $x_\bullet$ is obtained from $x'_\bullet$ by the same number of applications of $^\odot$. This is true because $(\beta^{-1}\alpha_m)_{m\in\mathbb{N}}$ always converges to 0 for $\beta \neq 0$, therefore it contains only finitely many elements outside R. For $\beta = 0$ the statement is trivial anyway.

Corollary 4.3 *Every torsion-free reduced Matlis-cotorsion module is division-free.*

Proof. For a torsion-free reduced Matlis-cotorsion module M assume $x \in UM$. Then by 3.3, x can be written as $x = \sum_{n=0}^{\infty} \alpha_n y_n$ with $\alpha_n y_n = 0$ for all $n \in \mathbb{N}$. Since M is torsion-free, this implies $y_n = 0$ for all $n \in \mathbb{N}$ with $\alpha_n \neq 0$. But this easily yields $x = 0$. □

Lemma 4.4 $\mathrm{Ext}^1(K/R, M) = \{0\}$ *holds for every divisible R-module M.*

Proof. Represent K as an increasing union of copies $\gamma_n^{-1}R$ of R. It suffices to show that every short exact sequence $0 \to M \to M' \to K/R \to 0$ splits. Since divisible modules are closed under extensions, M' is also divisible. Since the projection $q : M' \to K/R$ in this sequence is surjective, there is an $x_0 \in M'$ with $q(x_0) = \gamma_0^{-1} + R$. If x_n has already been defined with $q(x_n) = \gamma_n^{-1} + R$, the surjectivity of q yields an $y \in M'$ with $q(y) = \gamma_{n+1}^{-1} + R$, hence also $q((\gamma_n^{-1}\gamma_{n+1})y - x_n) = (\gamma_n^{-1}\gamma_{n+1})q(y) - q(x_n) = 0$. Therefore $(\gamma_n^{-1}\gamma_{n+1})y - x_n$ is in the kernel of q, i.e. in the image of the divisible module M'. Thus there is a $z \in M'$ with $q(z) = 0$ and $(\gamma_n^{-1}\gamma_{n+1})z = (\gamma_n^{-1}\gamma_{n+1})y - x_n$, i.e. $x_n = (\gamma_n^{-1}\gamma_{n+1})(y - z)$. Now we define $x_{n+1} := y - z \in M'$, thus $(\gamma_n^{-1}\gamma_{n+1})x_{n+1} = x_n$ and $q(x_{n+1}) = q(y) - q(z) = \gamma_{n+1}^{-1} + R$. Thus there is a unique linear map $K/R \to M'$, which maps $\gamma_n^{-1} + R$ to x_n for all $n \in \mathbb{N}$; this map splits the short exact sequence. □

For R a discrete valuation ring 4.4 is obvious because every divisible module is injective.

Theorem 4.5 *Mapping an arbitrary R-module M to the canonical map $M \to \mathrm{Ext}^1(K/R, M)$ yields the reflection from the category of R-modules to the category of reduced Matlis-cotorsion R-modules.*

Proof. First we claim that the canonical map $\mathrm{Ext}^1(K/R, M) \to \mathrm{Ext}^1(K/R, M/U^\infty M)$ is always an isomorphism. The long cohomology sequence for the functor $\mathrm{Hom}(K/R, -)$ applied to the short exact sequence $0 \to U^\infty M \to M \to M/U^\infty M \to 0$ contains the part $\mathrm{Ext}^1(K/R, U^\infty M) \to \mathrm{Ext}^1(K/R, M) \to \mathrm{Ext}^1(K/R, M/U^\infty M) \to \mathrm{Ext}^2(K/R, U^\infty M)$. The first module vanishes by 4.4, and the last part vanishes, since R has cohomological dimension 1. The assignment in the theorem is obviously a natural transformation. By 3.2 $U^\infty M$ is always in the kernel, so by 4.4 we can restrict our attention to reduced modules. Since K/R is a torsion module and $M/U^\infty M$ is reduced, we have $\mathrm{Hom}(K/R, M/U^\infty M) = \{0\}$. Now by 2.1 of [9] $\mathrm{Ext}^1(K/R, M)$ is a Matlis-cotorsion module.

For a reduced Matlis-cotorsion module M, the long cohomology sequence of $\mathrm{Hom}(-, M)$ applied to $0 \to R \to K \to K/R \to 0$ contains the part $\mathrm{Hom}(K, M) \to \mathrm{Hom}(R, M) \to \mathrm{Ext}^1(K/R, M) \to \mathrm{Ext}^1(K, M)$. Since the first and the last module vanish, the middle arrow is an isomorphism; thus also the canonical arrow $M \to \mathrm{Ext}^1(K/R, M/U^\infty M) \cong \mathrm{Ext}^1(K/R, M)$.

So we have a natural transformation from the identity functor to a functor that maps all R-modules to reduced Matlis-cotorsion R-modules, and for all reduced Matlis-cotorsion R-modules the natural transformation yields an isomorphism. Therefore this must be the reflection. □

5 The Chain of Ulm Functors

We have seen that every R-module M admitting infinitary linear combinations with (LC1) and (LC2) is *reduced*, but we have not seen that it is *division-free*, though we have not given a counterexample up to now. At first glance the difference looks quite harmless; we always have $U^\infty M = \{0\}$, but not necessarily $UM = \{0\}$. We do not see immediately why not all elements of UM have to be divisible, i.e. why we may have $U^2 \neq U^1$. Of course, the U^κ form a decreasing chain of functors; thus it must be eventually constant, and for reduced M this can happen only at $\{0\}$, i.e. there is a κ with $U^\infty M = U^\kappa M = \{0\}$. But the smallest κ with this property can be arbitrarily large, even for a reduced Matlis-cotorsion module. We shall see this below, using the machinery of infinitary linear combinations used above.

Theorem 5.1 *For every torsion-free reduced Matlis-cotorsion module M and for every ordinal κ there exists a reduced Matlis-cotorsion module P with $U^\kappa P \cong M$.*

Proof. We represent K as a union of an increasing sequence of $\alpha_n^{-1}R$, we assume $\alpha_0 = 1$ and we consider the set T of all tuples $t = ((\nu_1, \ldots, \nu_n), (m_1, \ldots, m_n))$, where the ν_k form a strictly decreasing chain of ordinals $< \kappa$, including the empty tuple Λ, and where $m_1, \ldots, m_n$ is a strictly increasing tuple of natural numbers; for such a pair of pairs we use the shorter notation $t = (\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)$, and we put $\nu_0 := \kappa$.

Now M^T is a torsion-free Matlis-cotorsion module; we write its elements as maps $\phi : T \to M$. Consider the submodule $M^{[T]} \subset M^T$ of all $\phi \in M^T$ such that for each $\gamma \in R \setminus \{0\}$ there are only finitely many $t \in T$ with $\phi(t) \notin \gamma M$; since R is powerful and M is torsion-free, hence also division-free, this implies $\phi(t) = 0$ for all but countably many $t \in T$; we write ϕ as a formally uncountable linear combination. For each $t \in T$ we have an R-module homomorphism $u_t : M \to M^{[T]}$ defined by $u_t(x)(t) := x$ for $x \in M$ and $u_t(x)(t') := 0$ for $t' \in T$ with $t' \neq t$. Let $N \subset M^{[T]}$ be the submodule of all infinitary linear combinations of elements of the form $u_{(\nu_1,\ldots,\nu_{n-1};m_1,\ldots m_{n-1})}(x) - (\alpha_{m_{n-1}}^{-1}\alpha_{m_n})u_{(\nu_1,\ldots,\nu_n;m_1,\ldots,m_n)}(x)$. In such a representation of a $\phi \in N$, we assume w.l.o.g. that all $t :=$

$(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)$ are different; otherwise we gather all entries with the same t. Moreover we can assume $x \neq 0$ for all summands; otherwise we can omit them. If $\phi \neq 0$, for some t the coefficient β of some $u_t(x)$ does not vanish; we can choose $t = (\nu_1, \ldots \nu_n; m_1 \ldots, m_n)$ in such a way that ν_n is minimal. Since M is torsion-free by hypothesis, this implies $\phi(t)(\beta\alpha_{m_{n-1}}^{-1}\alpha_{m_n})x \neq 0$. So if we have $\phi(t) = 0$ for all $t := (\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)$ with $\nu_n < \mu$ for some ordinal μ, we see that ϕ is a linear combination of generators t as above with $\nu_n \leq \mu$. Let $P := M^{[T]}/N$ be the quotient and let $q : M^{[T]} \to P$ be the canonical projection. Since $N \subset M^{[T]}$ is closed under infinitary linear combinations, P is a reduced Matlis-cotorsion module.

We claim that for each ordinal $\mu \leq \kappa$ the Ulm submodule $U^\mu P$ consists of all $q(x)$, where $x(\nu_1, \ldots, \nu_n; m_1, \ldots m_n) = 0$ holds whenever $\nu_n < \mu$. The limit step is obvious. Now we assume the statement for some $\mu < \kappa$ and prove it for $\mu + 1$. For the first direction consider a $\phi \in M^{[T]}$ with $\phi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n \leq \mu$ and a $\gamma \in R \setminus \{0\}$. By $\phi \in M^{[T]}$, the set $S := \{s \in T | \phi(s) \notin \gamma M\}$ is finite, and we have $\phi = \gamma\psi + \sum_{t \in T \setminus S} u_T(\phi(t))$ for some $\psi \in M^{[T]}$ with $\psi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n \leq \mu$; by induction hypothesis this implies $\psi \in U^\mu P$. Now there exists a natural number $l > m_n$ with $\alpha_{m_n}^{-1}\alpha_l \in \gamma R$. For $\nu_{n+1} := \mu$ and $m_{n+1} := l$ we get

$$u_{(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)}(x) - (\alpha_{m_n}^{-1}\alpha_{m_{n+1}})u_{(\nu_1, \ldots, \nu_{n+1}; m_1, \ldots, m_{n+1})}(x) \in N,$$

hence

$$q(u_{(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)}(x)) = (\alpha_{m_n}^{-1}\alpha_{m_{n+1}})q(u_{(\nu_1, \ldots, \nu_{n+1}; m_1, \ldots, m_{n+1})}(x)),$$

for all $x \in M$. By induction hypothesis we have

$$q(u_{(\nu_1, \ldots, \nu_{n+1}; m_1, \ldots, m_{n+1})}(x)) \in U^\mu P,$$

therefore

$$q(u_{(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)}(x)) \in (\alpha_{m_n}^{-1}\alpha_{m_{n+1}})U^\mu P \subset \gamma U^\mu P.$$

This implies

$$\phi = \gamma\psi + \sum_{s \in S} u_s(\phi(s)) \in \gamma U^\mu P,$$

proving $\phi \in UU^{\mu}P = U^{\mu+1}P$.

Conversely, assume $\phi \in M^{[T]}$ with $q(\phi) \in U^{\mu+1}P = UU^{\mu}P \subset U^{\mu}P$; by hypothesis we assume $\phi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n < \mu$. For every $\gamma \in R \setminus \{0\}$ there is a $\psi \in U^{\mu}P$ with $\psi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n < \mu$ and $q(\phi) = \gamma q(\psi) = q(\gamma\psi)$, hence $\phi - \gamma\psi \in N$. For $t = (\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) \in T$ we have $\phi - \gamma\psi(t) = 0$ whenever $\nu_n < \mu$. Then by our above considerations we can assume that only elements t with $\nu_n \geq \mu$ have non-zero coefficients; therefore we have $\phi - \gamma\psi(t) = 0$ for $\nu_n = \mu$. Thus we get $\phi(t) = \gamma\psi(t) \in \gamma M$; this proves $\phi(t) \in UM = \{0\}$, because M is division-free. □

This proof works only in the torsion-free case; otherwise the transfinite induction breaks down. We do not see whether 5.1 is still true otherwise, even in the division-free case. But of course, $U^{\kappa}P$ is not always division-free; the above construction also yields modules with prescribed division-free $U^{\kappa+1}P$. The question looks even interesting for $\kappa = 1$; then it would follow for all finite κ. Moreover, maybe one can only prove that every reduced Matlis-cotorsion module occurs as a submodule of some $U^{\kappa}P$.

Corollary 5.2 *The category of reduced Matlis-cotorsion R-modules has no cogenerator.*

Proof. Assume the contrary, i.e. let C be a cogenerator. Then there exists an ordinal κ with $U^{\infty}C = U^{\kappa}C = \{0\}$. Now by 5.1 there exists a reduced Matlis-cotorsion R-module M with $U^{\kappa}M = R \neq \{0\}$. Then an R-linear map $M \to C$ maps all elements of $U^{\kappa}M$ to 0, hence it does not separate them. □

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CENTRALIZERS IN ACTION ACCESSIBLE CATEGORIES

*by Dominique BOURN and George JANELIDZE**

Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract

Nous introduisons la notion de catégorie accessible qui comprend une grande partie des catégories protomodulaires, dont les catégories des groupes, des anneaux, des algèbres associatives et des algèbres de Lie. Cette notion a l'avantage de permettre de calculer intrinséquement les centralisateurs des sous-objets et des relations d'équivalence. Nous montrons que dans de telles catégories les notions de commutateurs pour les sous-objets et pour les relations d'équivalence coïncident.

We introduce and study action accessible categories. They provide a wide class of protomodular categories, including all varieties of groups, rings, associative and Lie algebras, in which it is possible to calculate centralizers of equivalence relations and subobjects. We show that, in those categories, the equivalence relation and subobject commutators agree with each other.

Key words : Protomodular and semi-abelian categories; centralizers; commutators; split exact sequences.
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Introduction.

When X is a subset of a group A, the centralizer $Z(X)$ of X in A is defined as

$$Z(X) = \{a \in A / x \in X \Rightarrow axa^{-1} = x\}$$

When X is a normal subgroup in A, sending $a \in A$ to the automorphism $c(a)$ of X defined by $c(a)(x) = axa^{-1}$ determines a group homomorphism $c : A \to Aut(X)$, and the centralizer $Z(X)$ can equivalently be defined as: $Z(X) = Ker(c)$.

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When X is a normal subobject in an object A in a semi-abelian category $\mathbb{C}$, the centralizer $Z(X)$ is to be defined as the largest subobject K of A with $[\![X, K]\!] = 0$ (here and below we write $[\![,]\!]$ for the "classical" *subobject commutator* in order to distinguish it from the *equivalence relation commutator* $[,]$). The existence of such a $Z(X)$ can then be proved in the case of a semi-abelian variety, but not in general: a counter-example was constructed by S. A. Huq [12].

Since the group $Aut(X)$ is a particular example of the *split extension classifier* (introduced in [3], and denoted there by $[X]$; see also [2]), it is natural to ask if the equality $Z(X) = Ker(c)$ still holds for the appropriate $c : A \to [X]$ whenever the protomodular category $\mathbb{C}$ is *action representable*, i.e. whenever the split extension classifier $[X]$ exists.

Among others, there are two main results in this paper:

(a) We not just answer positively the question above, but prove a stronger result applicable to a much wider class of categories, which we call *action accessible*. They include e.g. all varieties of groups, rings, associative and Lie algebras.

(b) As an application, we prove that in any action accessible category the equivalence relation and subobject commutators agree in the sense that $[R, S]$ =0 if and only if $[\![I_R, I_S]\!] = 0$, where I_R is the normal subobject associated with R; as we know from [6], this is also true in any strongly protomodular category, but for a very different reason.

The paper is divided into six sections as follows:

Section 1 introduces faithful split extensions and studies their simple properties, especially in the case of rings - which is the most important non-action-representable case. For familiar algebraic categories faithful split extensions

$$0 \longrightarrow X \overset{k}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

correspond to faithful actions of B on X, which is the reason of choosing the term "faithful". Note also that all *generic split extensions*

$$0 \longrightarrow X \rightarrowtail X \ltimes [X] \rightleftarrows [X] \longrightarrow 0$$

(in the sense of [3], with the concept of semidirect product introduced in [9]) are faithful by very definition.

Section 2 defines the *action accessible* categories as those "with access to faithful split extensions", i.e. as those where every split extension admits a morphism into a faithful one. It is shown that the following categories are action accessible:

- any action representable category (trivially);
- the category of rings;
- any Birkhoff subcategory of a *homological* action accessible category;
- the category of split epimorphisms into any object in an action accessible category.

Section 3 essentially shows that working with split extensions is the same as working with internal groupoids, and therefore allows to apply the constructions with split extensions to (internal) equivalent relations.

Section 4 shows, using the results of Section 3, how to calculate centralizers of equivalence relation as kernel pairs of morphisms into split extensions, and in particular concludes that all action representable and all action accessible (homological) categories admit centralizers.

Section 5 studies centralizers of normal subobjects, compares them with centralizers of equivalence relations and concludes that the equivalence relation and subobject commutators agree in any action accessible category.

Section 6 provides a new characterization of *antiadditivity* (=the property for an object of having trivial centres) via faithfulness of a particular split extension, extending a simple property of groups.
Remark: (a) The authors did their best to adjust the terminology and notation they use with those of the papers they refer to – even though in some cases it almost created disagreements They hope, however, that the choices they made will be most convenient for the readers, especially those who studied the book [1].
(b) The action accessibility defined in this paper has nothing to do with the concept of accessible category – it is only a coincidence of terminology.

1 Faithful split extensions

Let $\mathbb{C}$ be a finitely complete pointed category. Recall it is *protomodular* when for any diagram with $ps = 1_B$ and k the kernel of p:

$$0 \longrightarrow X \overset{k}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

the pair (k, s) is jointly strongly epic.

Now let $\mathbb{C}$ be a fixed pointed protomodular category, and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \overset{k}{\rightarrowtail} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \underset{l}{\rightarrowtail} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D & \longrightarrow & 0 \end{array} \qquad (1.1)$$

a diagram in $\mathbb{C}$, which has the following properties:
- it reasonably commutes, i.e. has $l = fk$, $qf = gp$, and $fs = tg$;
- it has $ps = 1_B$ and $qt = 1_D$;
- k and l are kernels of p and q respectively.

We will consider such a diagram as a morphism (g, f) of split extensions (with fixed X), write

$$(g, f) : (B, A, p, s, k) \to (D, C, q, t, l) \qquad (1.2)$$

and denote the category of such split extensions by

$$SplExt(X) = SplExt_{\mathbb{C}}(X)$$

The functor

$$SplExt(X) \to \mathbb{C} \qquad (1.3)$$

sending (B, A, p, s, k) to B is a faithful fibration in which every *vertical* morphism is an isomorphism (since $g = 1_D \Rightarrow f$ is an isomorphism) and therefore every morphism is *cartesian.* This follows from the protomodularity of $\mathbb{C}$, and, moreover, when $\mathbb{C}$ is just required to be pointed and to have finite limits, this is equivalent to protomodularity.

Observation 1.1. *Using protomodularity we observe:*
(a) to say that the functor (1.3) is faithful is of course the same as to say that the morphism f in (1.1) (provided it exists) is determined by other morphisms, which follows from the fact that the pair (k, s) is jointly (strongly) epic;
(b) to say that every morphism in $SplExt(X)$ is cartesian (with respect to the functor (1.3)) is the same as to say that in every diagram in $\mathbb{C}$ of the form (1.1) the square $qf = gp$ is a pullback;
(c) the category $SplExt(X)$ obviously has connected finite limits preserved by the functor (1.3), and, since this functor (1.3) is faithful, it not only preserves, but also reflects monomorphisms.

Definition 1.2. *An object in $SplExt(X)$ is said to be faithful, if any object in $SplExt(X)$ admits at most one morphism into it.*

Observation 1.3. *For arbitrary two objects X and Y in $\mathbb{C}$, consider the split extension $(Y, Y \times X, p_Y, i_Y, i_X)$, in which $p_Y : Y \times X \to Y$ is the product projection, and $i_Y = \langle 1, 0\rangle : Y \to Y \times X$ and $i_X = \langle 0, 1\rangle : X \to Y \times X$ are the "product injections". This split extension belongs to $SplExt(X)$, and it becomes its initial object if and only if $Y = 0$. Furthermore, it always admits a (unique) morphism into the initial object, and therefore it is faithful if and only if it is initial. In particular this implies that whenever the category $\mathbb{C}$ has at least one object X for which every object in $SplExt(X)$ is faithful, the category $\mathbb{C}$ has no non-zero objects, in other words $\mathbb{C}$ is indiscrete.*

Using Observation 1.1(c) we obtain:

Proposition 1.4. *For a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ with faithful codomain (D, C, q, t, l), the following conditions are equivalent:*
(a) (B, A, p, s, k) is faithful;
(b) (g, f) is a monomorphism in $SplExt(X)$;
(c) g is a monomorphism in $\mathbb{C}$.

In particular, the category $SplExt(X)$ might have a terminal object, which is to be called the *generic split extension with kernel X* (according to [3]), or the *universal split extension of X* (according to [2]); this is the case when $\mathbb{C}$ has *representable object actions* in the sense of [3] or is *action representative* in the sense of [2] (these two concepts coincide, except that the categories considered in [3] were required to be semi-abelian). The image of the generic split extension with kernel X under the functor (1.3) was called the split extension classifier for X and denoted by $[X]$ in [3], and by $D(X)$ in [2]. From Proposition 1.4 we obtain:

Corollary 1.5. *When $\mathbb{C}$ is action representative, the following conditions on an object (B, A, p, s, k) in $SplExt(X)$ are equivalent:*
(a) (B, A, p, s, k) is faithful;
(b) the corresponding classifying morphism $B \to [X]$ is a monomorphism.

Note that in the case of groups the morphism $B \to [X]$ becomes $B \to Aut(X)$, which justifies the term faithful. However there are other justification results beyond the action representative cases, such as Proposition 1.6 below or a similar result for commutative rings. When $\mathbb{C}$ is semi-abelian [13], the category $SplExt(X)$ is equivalent to the category of pairs (B, ξ), where ξ is an action of B on X in the sense of [9] (see [3] and [4] for details). Therefore Definition 1.2 in fact gives a definition of a *faithful object action*.

Proposition 1.6. *Let $\mathbb{C}$ be the variety* **Rg** *of (not-necessarily-unitary) rings, and X an object in $SplExt(X)$. Then the following conditions on an object (D, C, q, t, l) in $SplExt(X)$ are equivalent:*
(a) (D, C, q, t, l) is faithful;
(b) if d and d' are elements in D with $t(d)l(x) = t(d')l(x)$ and $l(x)t(d) = l(x)t(d')$ for all x in X, then $d = d'$;
(c) if d is an element in D with $t(d)l(x) = 0 = l(x)t(d)$ for all x in X, then $d = 0$.

Proof. $(b) \Leftrightarrow (c)$ is obvious.

$(a) \Rightarrow (b)$: for an object (D, C, q, t, l) in $SplExt(X)$ and an element d in D we can construct an object (B, A, p, s, k) in $SplExt(X)$ and a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ as follows:
• we take B to be the free algebra in $\mathbb{C}$ on a one-element set $\{z\}$;

• we define $g : B \to D$ as the unique ring homomorphism from B to D with $g(z) = d$;
• A is $B \times X$ as an abelian group, with the multiplication defined by

$$(b, x)(b', x') = (bb', bx' + xb' + xx'), \tag{1.4}$$

where bb' and xx' are defined as in B and in X respectively, and bx' and xb' are defined by

$$l(bx') = tg(b)l(x') \text{ and } l(xb') = l(x)tg(b') \tag{1.5}$$

respectively (using the fact that l is injective);
• we define p, s, k, and f by

$$p(b, x) = b, \; s(b) = (b, 0), \; k(x) = (0, x), \text{ and } f(b, x) = tg(b) + l(x) \tag{1.6}$$

respectively.

Checking that this determines a morphism in $SplExt(X)$ requires a long but straightforward calculation, which we omit. Let us now compare the morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ with the morphism $(g', f') : (B', A', p', s', k') \to (D, C, q, t, l)$ constructed in exactly the same way but with an element d' instead of d. We claim that if

$$t(d)l(x) = t(d')l(x) \text{ and } l(x)t(d) = l(x)t(d')$$

for all x in X, then $(B', A', p', s', k') = (B, A, p, s, k)$. Indeed, we observe:
• $B' = B$, $A' = A$ as abelian groups, and p', s', k' are the same maps as p, s, k respectively in any case. Therefore we only need to show that for all b, b' in B and x, x' in X, $(b, x)(b', x')$ in A' is the same as $(b, x)(b', x')$ in A.
• According to (1.4) and (1.5), to show that $(b, x)(b', x')$ in A' is the same as $(b, x)(b', x')$ in A for all b, b' in B and x, x' in X, it suffices to show that:

$$tg(b)l(x) = tg'(b)l(x) \text{ and } l(x)tg(b) = l(x)tg'(b) \tag{1.7}$$

for all b in B and x in X.
• Since, by the assumption on d and d', the equalities (1.7) hold for

$b = z$, it suffices to show that the set of elements b in B for which the equalities (1.7) hold form a subring in B. Moreover, since that set is obviously a subgroup of the additive group of B, we only need to show that it is closed under the multiplication in B. This, however, easily follows from the fact that tg and tg' are ring homomorphisms and the multiplication in D is associative.

Next, since (D, C, q, t, l) is faithful, $(B', A', p', s', k') = (B, A, p, s, k)$ implies $g = g'$, and so $d = d'$.

$(b) \Rightarrow (a)$: let (g, f) and (g', f') be morphisms from (B, A, p, s, k) to (D, C, q, t, l) and b and x be elements in B and X respectively. Since $p(s(b)k(x)) = ps(b)pk(x) = 0$, there exists y in X with $k(y) = s(b)k(x)$, and we have:

$$tg(b)l(x) = fs(b)fk(x) = f(s(b)k(x)) = fk(y) = l(y) = f'k(y)$$

$$= f'(s(b)k(x)) = f's(b)f'k(x) = tg'(b)l(x)$$

and similarly $l(x)tg(b) = l(x)tg'(b)$. Condition (b) then tells us that $g(b) = g'(b)$ for all b in B. That is, $g = g'$, and since k and s are jointly epic this also gives $f = f'$, as desired. □

2 Action accessibility

Definition 2.1. *Let $\mathbb{C}$ be a pointed protomodular category. An object in SplExt(X) is said to be accessible, if it admits a morphism into a faithful object. If every object in SplExt(X) is accessible, we will say that $\mathbb{C}$ is action accessible.*

As immediately follows from this definition, *every action representative category* is *action accessible*. So this is in particular the case for the categories **Gp** of groups and R-**Lie** of Lie R-algebras. The following example of action accessible category will show that the converse is not true:

Proposition 2.2. *The variety* **Rg** *of (not-necessarily-unitary) rings is action accessible.*

Proof. For an object (B, A, p, s, k) in $SplExt(X)$ we construct the desired morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ into a faithful object as follows:

• The set $I = \{b \in B \mid \forall_{x \in X}\ s(b)k(x) = 0 = k(x)s(b)\}$ is an ideal in B. Indeed, for b in I and b' in B we have $s(b'b)k(x) = s(b')s(b)k(x) = 0 = s(b)s(b')k(x) = s(bb')k(x)$, where the third equality follows from the fact that $s(b')k(x) = k(y)$ for some y in X; similarly $k(x)s(b'b) = 0 = k(x)s(bb')$. We take $D = B/I$.

• The image $s(I)$ of I under s is an ideal in A. In order to prove this, it suffices to show that for b in I, b' in B, and x in X, the elements $s(b')s(b)$, $s(b)s(b')$, $k(x)s(b)$, and $s(b)k(x)$ are in $s(I)$. For the elements $s(b')s(b) = s(b'b)$ and $s(b)s(b') = s(bb')$ this follows from the fact that I is an ideal in B. The elements $k(x)s(b)$ and $s(b)k(x)$ are simply equal to 0 by definition of I. We take $C = A/s(I)$.

• We define q, t, and l as the morphisms induced by p, s, and k respectively, and take f and g to be the canonical morphisms $A \to A/s(I)$ and $B \to B/I$. This obviously determines a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$, since all the maps involved are ring homomorphisms and the resulting diagram considered as a diagram in the category of abelian groups becomes isomorphic to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \rightarrowtail & X \oplus B & \rightleftarrows & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \rightarrowtail & X \oplus B/I & \rightleftarrows & B/I & \longrightarrow & 0 \end{array}$$

of canonical morphisms.

It remains to prove that (D, C, q, t, l) is faithful. According to Proposition 1.6 it suffices to prove that if d is an element in D with $t(d)l(x) = 0 = l(x)t(d)$ for all x in X, then $d = 0$. We have $d = b + I$ for some b in B, and then $t(d)l(x) = 0 = l(x)t(d)$ in D means that the elements $s(b)k(x)$ and $k(x)s(b)$ are in $s(I)$. On the other hand, $d = 0$ in D means that b is in I, i.e. that $s(b)k(x) = 0 = k(x)s(b)$ for all x in X. That is, we have to prove the implication

$$\forall_{x \in X}\ s(b)k(x), k(x)s(b) \in s(I) \Rightarrow \forall_{x \in X}\ s(b)k(x) = 0 = k(x)s(b)$$

However, it follows from the much stronger and obvious implication

$$s(b)k(x), k(x)s(b) \in s(B) \Rightarrow s(b)k(x) = 0 = k(x)s(b)$$

□

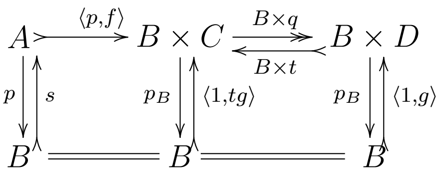
Many other examples of action accessible categories can be obtained from

Proposition 2.3. *If $\mathbb{C}$ is an action accessible homological (i.e. pointed protomodular regular) category and $\mathbb{D}$ is a Birkhoff subcategory in $\mathbb{C}$, then $\mathbb{D}$ also is action accessible.*

Proof. For X in $\mathbb{D}$ and a morphism $(g,f):(B,A,p,s,k)\to(D,C,q,t,l)$ with (B,A,p,s,k) in $SplExt_{\mathbb{D}}(X)$ and a faithful object (D,C,q,t,l) in $SplExt_{\mathbb{C}}(X)$ just take ($\mathbb{C}$ being regular) $(g',f'):(B,A,p,s,k)\twoheadrightarrow(D',C',q',t',l')$, where (D',C',q',t',l') is the suitably constructed image of (g,f), and (g',f') is induced by (g,f). □

Now let $Pt_{\mathbb{C}}(Y)$ denote the category whose objects are the split epimorphisms above Y and morphims are the commutative triangles between those split epimorphims. When $\mathbb{C}$ is protomodular, then $Pt_{\mathbb{C}}(Y)$ is pointed protomodular.

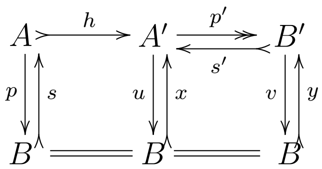
Proposition 2.4. *Let $\mathbb{C}$ be a pointed protomodular category.*
(a) Given a morphism $(g,f):(B,A,p,s,k)\to(D,C,q,t,l)$ in the category $SplExt(X)$, the diagram



(2.1)

is a split extension in $Pt_{\mathbb{C}}(B)$ that is faithful whenever so is (D,C,q,t,l).
(b) If $\mathbb{C}$ is action accessible, then, for any object B, the category $Pt_{\mathbb{C}}(B)$ is action accessible.

Proof. (a): Omitting straightforward verification of the first assertion, consider another split extension



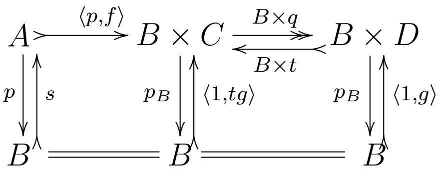
(2.2)

with the same A, p, and s, and two morphisms (m, n) and (m', n') from it to the split extension (2.1). We have to prove that $n = n'$. Since $p_B n = v = p_B n'$, it suffices to prove that the composites $p_D n$ and $p_D n'$ of n and n' with the projection $p_D : B \times D \to D$ are equal to each other. This, however, follows from the fact that $(p_C m, p_D n)$ and $(p_C m', p_D n')$ can be presented as two parallel morphisms into (D, C, q, t, l) in $SplExt_{\mathbb{C}}(X)$ and (D, C, q, t, l) is faithful.

(b): Let us now begin with an arbitrary split extension (2.2) in $Pt_{\mathbb{C}}(B)$, and let (X, k') be the kernel of p'. Since $\mathbb{C}$ is action accessible, there is a faithful split extension (D, C, q, t, l) and a morphism

$$(g', f') : (B', A', p', s', k') \to (D, C, q, t, l)$$

in $SplExt_{\mathbb{C}}(X)$. After that all we need is to observe that the morphism (g', f') induces a morphism from the split extension (2.2) to the split extension



constructed as follows:

• putting $g = g'y$ makes $B \times q$ a morphism $(B \times C, p_B, \langle 1, tg \rangle) \to (B \times D, p_B, \langle 1, g \rangle)$ in the category $Pt_{\mathbb{C}}(B)$, and we define $k : (A, p, s) \to (B \times C, p_B, \langle 1, tg \rangle)$ as the kernel of that morphism;

• we then define $f : A \to C$ as the composite of k with the product projection $B \times C \to C$, which makes $k = \langle p, f \rangle$;

- and that this split extension is faithful by (a). □

3 The fibration of X-groupoids

In this section we extend the previous observations to internal reflexive graphs and groupoids, which we shall need to introduce centralizers. For an object X in $\mathbb{C}$, by a reflexive graph structure on an object (B, A, p, s, k) in $SplExt(X)$ we will mean a morphism $u : A \to B$ with $us = 1_B$; we will then also say that (B, A, p, s, u) is the underlying reflexive graph of $((B, A, p, s, k), u)$. Conversely, given any reflexive graph

(B, A, d_0, s_0, d_1), the morphism d_1 gives a reflexive graph structure on the object (B, A, d_0, s_0, k) in $SplExt(X)$, where (X, k) is any kernel of d_0.

Since every protomodular category is a Maltsev category, being an internal groupoid in $\mathbb{C}$ is the same as being an internal reflexive graph in $\mathbb{C}$ satisfying certain property (not having an additional structure). Specifically, a reflexive graph (B, A, d_0, s_0, d_1) is a groupoid if and only if the commutator

$$[R[d_0], R[d_1]] \tag{3.1}$$

is trivial, where $R[f]$ denotes (as in [1]) the equivalence relation determined by the kernel pairs of any morphism f:

$$R[f] \underset{p_1}{\overset{\overset{p_0}{\longrightarrow}}{\underset{\longrightarrow}{\overset{s_0}{\longleftarrow}}}} X \xrightarrow{\;f\;} Y$$

and where, given any pair (R, S) of equivalence relations on an object X, we say that the commutator $[R, S]$ is trivial and write $[R, S] = 0$ when the pair (R, S) has a connector [7], i.e. it admits a morphism:

$$p : R \times_X S \to X,$$

which, written with generalized elements as $(xRySz) \mapsto p(x, y, z)$, satisfies the identities $p(x, y, y) = x$ and $p(y, y, z) = z$.

Accordingly, by a groupoid structure on an object (B, A, p, s, k) in $SplExt(X)$ we will mean a morphism $u : A \to B$ for which $us = 1_B$ and $[R[p], R[u]] = 0$; the system (B, A, p, s, k, u) will then be called an X-groupoid. X-groupoids form a category $Grpd(X) = Grpd_{\mathbb{C}}(X)$, in which a morphism

$$(g, f) : (B, A, p, s, k, u) \to (D, C, q, t, l, v) \tag{3.2}$$

is a morphism $(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$ in $SplExt(X)$ with $vf = gu$. Similarly to the functor (1.3), the functor

$$Grpd(X) \to \mathbb{C} \tag{3.3}$$

sending (B, A, p, s, k, u) to B is a faithful fibration in which every vertical morphism is an isomorphism and therefore every morphism is cartesian. This last point means that every morphism in $Grpd(X)$ determines a discrete fibration of (internal) groupoids.

Similarly to Definitions 1.2 and 2.1, we introduce

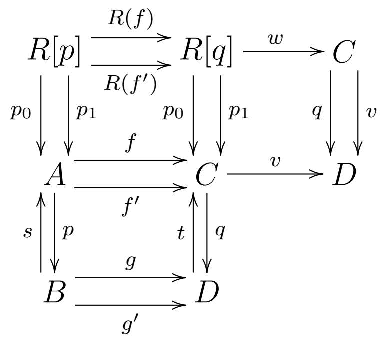
Definition 3.1. *(a) An X-groupoid is said to be faithful, if any X-groupoid admits at most one morphism into it.*
(b) An X-groupoid is said to be accessible, if it admits a morphism into a faithful X-groupoid. If every X-groupoid is accessible, we will say that $\mathbb{C}$ is groupoid accessible.

Lemma 3.2. *An X-groupoid is faithful if and only if its underlying object of $SplExt(X)$ is faithful.*

Proof. Let (D, C, q, t, l, v) be a faithful X-groupoid, and

$$(g, f), (g', f') : (B, A, p, s, k) \rightrightarrows (D, C, q, t, l)$$

a pair of morphisms in $SplExt(X)$. Consider the diagram



where the top parts of the first two columns the kernel equivalence relations of p and q, the top morphisms between them are induced by (g, f) and (g', f'), and w is the "division map" (with generalized elements it would be written as $w(\phi, \psi) = \psi\phi^{-1}$) of the kernel equivalence relation of q considered as a groupoid. Since the X-groupoid (D, C, q, t, l, v) is faithful, we get $vf = vf'$. Thus we have $g = vtg = vfs = vf's' = g'$.

Conversely, let (D, C, q, t, l, v) be a X-groupoid with a faithful underlying action. Then any pair

$$(g, f), (g', f') : (B, A, p, s, k, u) \rightrightarrows (D, C, q, t, l, v)$$

of morphisms in $Grpd(X)$ determines an underlying pair of morphisms in $SplExt(X)$, and consequently $g = g'$. □

By this lemma and Proposition 5.1 in [2] which shows that any split extension classifier underlies an internal groupoid, *any action representative category* $\mathbb{C}$ *is groupoid accessible.* We then get the following:

Proposition 3.3. *Suppose the category* $\mathbb{C}$ *is pointed protomodular and groupoid accessible; then it is action accessible.*

Proof. Just observe that any object (B, A, p, s, k) in $SplExt(X)$ admits a morphism into the underlying object of an X-groupoid, e.g. of the kernel equivalence relation of p, and use the previous lemma. □

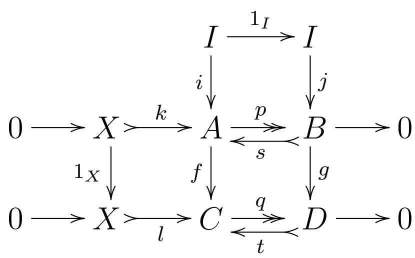
On the other hand we have:

Proposition 3.4. *Suppose* $\mathbb{C}$ *is homological. Let*

$$(g, f) : (B, A, p, s, k) \to (D, C, q, t, l)$$

be a morphism in $SplExt(X)$ *in which* g *(and therefore also* f*) is a normal epimorphism. When* (B, A, p, s, k) *has a reflexive graph structure or a groupoid structure* u*, the object* (D, C, q, t, l) *also has such a structure* v *with* $vf = gu$.

Proof. Consider the diagram



(3.4)

in which (I, i) and (I, j) are the kernels of f and g respectively, and we can assume that they involve the same object I and have $j = pi$ (and $i = sj$) since the square formed by $qf = gp$ is a pullback. Given a morphism $u : A \to B$ with $us = 1_B$, we observe:

- Since f being a normal epimorphism is a cokernel of i, and since

$gui = gusj = gj = 0$, there exists a morphism $v : C \to D$ with $vf = gu$. Moreover, for such a morphism v, we have $vtg = vfs = gus = g$, and since g is an epimorphism, we obtain $vt = 1_D$.
• After this it remains to prove that if u is a groupoid structure on (B, A, p, s, k), and v is a reflexive graph structure on (D, C, q, t, l) with $vf = gu$, it is also a groupoid structure. But this is the case by Theorem 3.1 in [10], since any homological category is Malt'sev and regular, and g, f are both normal (and thus regular) epimorphisms. □

From this proposition, using also Proposition 1.4 and the obvious (normal epi, mono)-factorization system in $SplExt(X)$, we obtain:

Corollary 3.5. *Suppose $\mathbb{C}$ is homological. Then $\mathbb{C}$ is action accessible if and only if it is is groupoid accessible.*

4 The centralizer of an accessible equivalence relation

Here is our main result:

Theorem 4.1. *Let R be an equivalence relation on an object B in a protomodular category $\mathbb{C}$, X an object in $\mathbb{C}$, and $(g, f) : (B, A, p, s, k, u) \to (D, C, q, t, l, v)$ a morphism in $Grpd(X)$, in which (B, A, p, s, k, u) is the equivalence relation R considered as a groupoid in $\mathbb{C}$, and (D, C, q, t, l, v) is faithful. Then the kernel pair $R[g]$ of g is the centralizer of R, i.e. the largest equivalence relation on B with $[R, R[g]] = 0$.*

Proof. The fact the commutator $[R, R[g]]$ is trivial follows from the fact that the morphism $R[f] \to R[g]$ of equivalence relations induced by (p, q) is a discrete fibration of groupoids (see [7] for details).
It remains to prove that if an equivalence relation $R' = (B, A', p', s', u')$ has $[R, R'] = 0$, then R' is less or equal to the kernel pair of g. That is,

we have to prove that $gp' = gu'$ whenever there exists a diagram

$$\begin{array}{ccc} E & \overset{e'_1}{\underset{e'_2}{\rightrightarrows}} & A' \\ {\scriptstyle e_2}\downarrow\;\downarrow{\scriptstyle e_1} & & {\scriptstyle u'}\downarrow\;\downarrow{\scriptstyle p'} \\ A & \overset{p}{\underset{u}{\rightrightarrows}} & B \end{array} \tag{4.1}$$

in which each pair of parallel arrows determines an equivalence relation, and the pairs (e_1, p') and (e_2, u') determine discrete fibrations. Since the relevant squares are pullbacks, the two horizontal top arrows in (4.1) determine an object $Grpd(X)$, and both pairs (e_1, p') and (e_2, u') determine a morphism from that object to (B, A, p, s, k, u). Composing these morphisms with (g, f) and using the fact that (D, C, q, t, l, v) is faithful, we obtain the desired equality. □

Corollary 4.2. *All equivalence relations in a groupoid accessible category $\mathbb{C}$ have centralizers. This is the case in particular for any action representative category and any homological action accessible category.*

5 Centralizer of subobjects and centralizer of equivalence relations

As soon as the category $\mathbb{C}$ is pointed protomodular, there is an intrinsic notion of commutation for subobjects; see [5]. Indeed, given any pair (X, Y) of objects, the following downward square is a pullback:

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X \times Y \\ \downarrow & & {\scriptstyle p_Y}\downarrow\uparrow{\scriptstyle i_Y} \\ 0 & \longrightarrow & Y \end{array}$$

and consequently the pair (i_X, i_Y) is jointly strongly epic. Accordingly given any pair of subobject $x : X \rightarrowtail Z$, $y : Y \rightarrowtail Z$, there is at most one map $\phi : X \times Y \to Z$ such that $\phi i_X = x$ and $\phi i_Y = y$. When this is the case, we say that the subojects commute, call ϕ the cooperator of these two subobjects, and write $[\![X, Y]\!] = 0$ as in [6].

On the other hand, since $\mathbb{C}$ is pointed, any equivalence relation R on Z determines a subobject, namely the "equivalence class" $i_R = d_1 k : I_R \rightarrowtail Z$ of the initial map given by the following pullback:

$$\begin{array}{ccccc} I_R & \xrightarrow{k} & R & \xrightarrow{d_1} & Z \\ \downarrow & & \downarrow{\scriptstyle d_0} & & \\ 0 & \xrightarrow[\alpha_Z]{} & Z & & \end{array}$$

The following lemma is an obvious consequence of protomodularity:

Lemma 5.1. *The normalization function which associates with any equivalence relation on Z its normal subobject*

$$Rel(Z) \to Sub_Z,\ R \mapsto I_R$$

preserves and reflects the order.

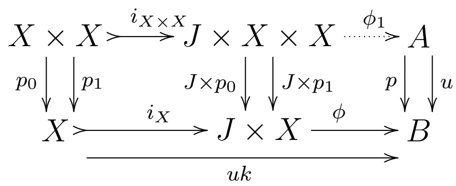
It is also clear that $[R, S] = 0$ implies $[\![I_R, I_S]\!] = 0$. The converse is true for strongly protomodular categories, but not in general, as shown in [6]. We are now going to show that groupoid accessible categories share this converse property with the strongly protomodular ones.

For, let us begin with the following observation. Let R be an equivalence relation on an object B, whose normalization is X and the corresponding X-groupoid is (B, A, p, s, k, u). When $\mathbb{C}$ is groupoid accessible, there is a morphism $(g, f) : (B, A, p, s, k, u) \to (D, C, q, t, l, v)$ in $Grpd(X)$ with the groupoid (D, C, q, t, l, v) being faithful. We have shown that the kernel pair $R[g]$ of g is the centralizer of R, i.e. the largest equivalence relation on B that commutes with R.

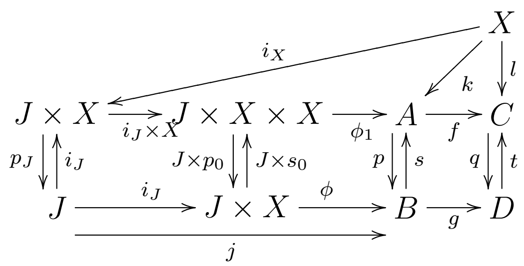
Proposition 5.2. *Suppose $\mathbb{C}$ is groupoid accessible. For R and g as above, the kernel morphism $k_g : K_g \rightarrowtail B$ of g (which is the normalization of $R[g]$ as well) is the largest subobject of B commuting with the normalization $uk : X \rightarrowtail B$ of R.*

Proof. Of course $[R, R[g]] = 0$ implies $[\![X, K_g]\!] = 0$; see [1] for instance. Suppose now we have any momomorphism $j : J \rightarrowtail B$ such that $[\![X, J]\!] = 0$; we have to check that J is less or equal to K_g, which is nothing but $gj = 0$. For, we will construct various morphisms of split

extensions and then use Observation 1.3 as follows:
• Let $\phi : J \times X \to B$ be the cooperator of $j : J \rightarrowtail B$ and $uk : X \rightarrowtail B$. First we construct the diagram



where: (a) the left-hand square is a discrete fibration of equivalence relations; (b) since every such discrete fibration (="fibrant morphism") in a protomodular category is cocartesian with respect to the forgetful functor into the ground category (see Lemma 5.1 in [8] or Lemma 6.1.6 in [1]), the new morphism ϕ_1 can be defined as the morphism making the right-hand side of the diagram an internal functor.
• Note that $\phi_1 i_{X\times X} i_1 = k$, since $p\phi_1 i_{X\times X} i_1 = ukp_0 i_1 = 0 = pk$, $u\phi_1 i_{X\times X} i_1 = ukp_1 i_1 = uk$, and p and u are jointly monic.
• Next, using the morphism ϕ_1 above, we construct the diagram



(in obvious notation), which reasonably commutes, i.e.: (a) its top part commutes; (b) its bottom part formed by solid arrows represent morphisms between split epimorphisms (with specified splittings). Accordingly we have a morphism in $SplExt(X)$:

$$(gj, f\phi_1(i_J \times X)) : (J, J \times X, p_J, i_J, i_X) \to (D, C, q, t, l)$$

• Since (D, C, q, t, l) is faithful, Observation 1.3 then tell us that $gj = 0$. $\square$

Observation 5.3. *If (g, f) is the unique morphism from the indiscrete relation $\nabla_X = (X, X \times X, p_0, s_0, p_1)$ into a faithful groupoid, then $R[g]$ is the largest equivalence relation commuting with ∇_X and K_g is nothing but the largest subobject commuting with 1_X, namely the centre ZX of X.*

Remark: When the category $\mathbb{C}$ in question is the category **Rg** of rings, this centre ZX is nothing but the *annihilator* of the ring X.

Theorem 5.4. *Suppose $\mathbb{C}$ is groupoid accessible. Let R and S be two equivalence relations on an object X. Then $[R, S] = 0$ if and only if $[\![I_R, I_S]\!] = 0$.*

Proof. We have already noticed that $[R, S] = 0$ implies $[\![I_R, I_S]\!] = 0$, which is a very general fact. Conversely suppose $[\![I_R, I_S]\!] = 0$. So, by Proposition 5.2, we have $I_S \subset K_g$, and according to Lemma 5.1, we have also $S \subset R[g]$. Whence, according to Theorem 4.1, $[R, S] = 0$. $\square$

6 A characterization of antiadditivity

Recall that a morphism $k : X \to A$ is said to be *central*, if there is a (necessarily unique) cooperator $\phi : X \times A \to A$ such that $\phi i_X = k$ and $\phi i_A = 1_A$; we use here the terminology of [5] again, although this concept of centrality (and of commutator) was originally studied by S. A. Huq [11] (in a slightly different context). In accordance with the terminology of [5], let us call an object A *antiadditive* if there are no nonzero central morphisms into it; that is, a pointed protomodular category is antiadditive in the sense of [5], see also [1], if and only if every object in it is antiadditive in our sense. If $\mathbb{C}$ is antiadditive, any abelian object is trivial. When the ground category $\mathbb{C}$ is homological, an object A is antiadditive if and only if *A has a trivial centre*, and $\mathbb{C}$ is antiadditive if and only if *$\mathbb{C}$ has no non trivial abelian objects.*

Theorem 6.1. *An object A in a pointed protomodular category $\mathbb{C}$ is antiadditive if and only if the split extension*

$$0 \longrightarrow A \xrightarrow{\langle 0,1 \rangle} A \times A \underset{s_0}{\overset{p_0}{\rightleftarrows}} A \longrightarrow 0 \tag{6.1}$$

is faithful.

Proof. Suppose the split extension (6.1) is faithful and consider any central morphism $k : X \to A$ with cooperator ϕ. Since both

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{i_A}{\rightarrowtail} & X \times A & \underset{i_X}{\overset{p_X}{\rightleftarrows}} & X & \longrightarrow & 0 \\ & & {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle \langle kp_X, \phi\rangle} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & A & \underset{\langle 0,1\rangle}{\rightarrowtail} & A \times A & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & A & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{i_A}{\rightarrowtail} & X \times A & \underset{i_X}{\overset{p_X}{\rightleftarrows}} & X & \longrightarrow & 0 \\ & & {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle 0\times 1_A} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & A & \underset{\langle 0,1\rangle}{\rightarrowtail} & A \times A & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & A & \longrightarrow & 0 \end{array}$$

are morphisms in $SplExt(A)$, we obtain $k = 0$.

Conversely, suppose A antiadditive and suppose we have a morphism (k, f) in $SplExt(A)$ whose codomain is the split extension (6.1). Then, since the square formed by f, k, and the appropriate arrows between them is a pullback, the domain of (k, f) must be isomorphic to the split extension

$$0 \longrightarrow A \overset{i_A}{\rightarrowtail} X \times A \underset{\langle 1_X, k\rangle}{\overset{p_X}{\rightleftarrows}} X \longrightarrow 0 \tag{6.2}$$

Therefore two parallel morphisms (k, f) and (l, g) into the split extension (6.1) will create an isomorphism $h : X \times A \to X \times A$ with $p_X h = p_X$, $h\langle 1_X, k\rangle = \langle 1_X, l\rangle$, and $hi_A = i_A$. Composing h with the projection $p_A : X \times A \to A$ we then obtain a morphism $\phi : X \times A \to A$ with $\phi\langle 1_X, k\rangle = l$ and $\phi i_A = 1_A$. The second identity makes ϕi_X central, and so $\phi i_X = 0$. Together with $\phi i_A = 1_A$ this implies $\phi = p_A$, and then $l = \phi\langle 1_X, k\rangle = p_A\langle 1_X, k\rangle = k$. Therefore the split extension (6.1) as faithful, as desired. □

Corollary 6.2. *A pointed protomodular category* $\mathbb{C}$ *is antiadditive if and only if the split extension (6.1) is faithful for each object* A *in* $\mathbb{C}$*.*

Let us now assume that the category $\mathbb{C}$ is action representable and call the morphism $A \to [A]$ corresponding to the split extension (6.1) *canonical*. From the previous results and Corollary 1.5 we obtain:

Corollary 6.3. *Let $\mathbb{C}$ be a (pointed protomodular) action representative category. Then:*
(a) an object A in $\mathbb{C}$ is antiadditive if and only if the canonical morphism $A \to [A]$ is a monomorphism;
(b) the category $\mathbb{C}$ is antiadditive if and only if the canonical morphism $A \to [A]$ is a monomorphism for each object A in $\mathbb{C}$.

Note that:
• Corollary 6.3(a) applied to the category of groups becomes the following obvious and yet nice observation: a group A has trivial centre if and only if the canonical homomorphism $A \to Aut(A)$ is injective.
• In several action representative categories, such as the dual $\mathbf{Set}_*^{\mathrm{op}}$ of the category of pointed sets, or the categories **BooRg** and **vNRg** of Boolean rings and von Neumann regular rings, the canonical morphisms $A \to [A]$ have been independently shown to be monomorphisms for all object A. As we see now, this can be used as a proof of their antiadditivity – even though direct proofs are also easy.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. L-3 (2009)*

ARE ALL COFIBRANTLY GENERATED MODEL CATEGORIES COMBINATORIAL?

*by J. ROSICKÝ**

Dedicated to Francis Borceux at the occasion of his sixtieth birthday

Abstract. G. Raptis has recently proved that, assuming Vopěnka's principle, every cofibrantly generated model category is Quillen equivalent to a combinatorial one. His result remains true for a slightly more general concept of a cofibrantly generated model category. We show that Vopěnka's principle is equivalent to this claim. The set-theoretical status of the original Raptis' result is open.

Résumé. G. Raptis a récemment démontré que, sous le principe de Vopěnka, chaque catégorie de modèles à engendrement cofibrant est Quillen équivalente à une catégorie de modèles combinatoire. Son résultat est valable pour un concept un peu plus général de catégorie de modèles à engendrement cofibrant. On va démontrer que le principe de Vopěnka est équivalent à cette assertion. Le statut ensembliste du résultat de Raptis est ouvert.

Keywords: cofibrantly generated model category, combinatorial model category, Vopenka's principle.
Mathematics Subject Classifications: 55U35, 18G55.

Combinatorial model categories were introduced by J. H. Smith as model categories which are locally presentable and cofibrantly generated. There are of course cofibrantly generated model categories which are not combinatorial – the first example is the standard model category of topological spaces. This model category is Quillen equivalent to the combinatorial model category of simplicial sets. G. Raptis [6] has recently proved a somewhat surprising result saying that, assuming Vopěnka's principle, every cofibrantly generated model category is Quillen equivalent to a combinatorial model category. Vopěnka's principle is a set-theoretical axiom implying the existence of very large cardinals (see [2]). A natural question is whether Vopěnka's principle (or other set theory) is needed for Raptis' result.

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A *model category* is a complete and cocomplete category $\mathcal{M}$ together with three classes of morphisms $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ called *fibrations*, *cofibrations* and *weak equivalences* such that

(1) $\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts in the arrow category $\mathcal{M}^{\to}$, and

by Jiri ROSICKY

(2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Morphisms from $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations* while morphisms from $\mathcal{C} \cap \mathcal{W}$ *trivial cofibrations*.

A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{M}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{M}$ such that

(1) $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$, and
(2) any morphism h of $\mathcal{M}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

Here, $\mathcal{L}^{\square}$ consists of morphisms having the right lifting property w.r.t. each morphism from $\mathcal{L}$ and ${}^{\square}\mathcal{R}$ consists of morphisms having the left lifting property w.r.t. each morphism from $\mathcal{R}$.

The standard definition of a cofibrantly generated model category (see [5]) is that the both weak factorization systems from its definition are cofibrantly generated in the following sense. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated if there exists a set $\mathcal{X}$ of morphisms such that

(1) the domains of $\mathcal{X}$ are small relative to $\mathcal{X}$-cellular morphisms, and
(2) $\mathcal{X}^{\square} = \mathcal{R}$.

Here, $\mathcal{X}$-cellular morphisms are transfinite compositions of pushouts of morphisms of $\mathcal{X}$. The consequence of this definition is that $\mathcal{L}$ is the smallest cofibrantly closed class containing $\mathcal{X}$. A cofibrantly closed class is defined as a class of morphisms closed under transfinite compositions, pushouts and retracts in $\mathcal{M}^{\to}$. Moreover, one does not need to assume that $(\mathcal{L}, \mathcal{R})$ is a weak factorization system because it follows from (1) and (2). This observation led to the following more general definition of a cofibrantly generated weak factorization system (see [1]).

A weak factorization system $(\mathcal{L}, \mathcal{R})$ is *cofibrantly generated* if there exists a set $\mathcal{X}$ of morphisms such that $\mathcal{L}$ is the smallest cofibrantly closed class containing $\mathcal{X}$. The consequence is that $\mathcal{X}^{\square} = \mathcal{R}$. A model category is *cofibrantly generated* if the both weak factorization systems from its definition are cofibrantly generated in the new sense. It does not affect the definition of a combinatorial model category because all objects are small in a locally presentable category. Moreover, the proof of Raptis [6] works for cofibrantly generated model categories in this sense as well.

We will show that Vopěnka's principle follows from the fact that every cofibrantly generated model category (in the new sense) is Quillen equivalent to a combinatorial model category. We do not know whether this is true for standardly defined cofibrantly generated model categories as well. Our proof uses the trivial model structure on a category $\mathcal{M}$ where all morphisms are cofibrations and weak equivalences are isomorphisms.

Given a small full subcategory $\mathcal{A}$ of a category $\mathcal{K}$, the *canonical functor*

$$E_{\mathcal{A}}\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

assigns to each object K the restriction

$$E_{\mathcal{A}}K = \hom(-,K)\big/\mathcal{A}^{\mathrm{op}}$$

of its hom-functor $\hom(-,K)\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ to $\mathcal{A}^{\mathrm{op}}$ (see [2], 1.25).

A small full subcategory $\mathcal{A}$ of a category $\mathcal{K}$ is called *dense* provided that every object of $\mathcal{K}$ is a canonical colimit of objects from $\mathcal{A}$. It is equivalent to the fact that the canonical functor

$$E_{\mathcal{A}}\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

is a full embedding (see [2], 1.26). A category $\mathcal{K}$ is called *bounded* if it has a (small) dense subcategory (see [2]).

Dense subcategories were introduced by J. R. Isbell [4] and called left adequate subcategories. The following result is easy to prove and can be found in [4].

Lemma 1. *Let $\mathcal{A}$ be dense subcategory of $\mathcal{K}$ and $\mathcal{B}$ a small full subcategory of $\mathcal{K}$ containing $\mathcal{A}$. Then $\mathcal{B}$ is dense.*

Proposition 2. *Let $\mathcal{K}$ be a cocomplete bounded category. Then $(\mathcal{K}, \mathrm{Iso})$ is a cofibrantly generated weak factorization system.*

Proof. Clearly, $(\mathcal{K}, \mathrm{Iso})$ is a weak factorization system. The canonical functor

$$E_{\mathcal{A}}\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

has a left adjoint F (see [2], 1.27). The weak factorization system

$$(\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}, \mathrm{Iso})$$

in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is cofibrantly generated (see [9], 4.6). Thus there is a small full subcategory $\mathcal{X}$ of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ such that each morphism in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a retract of a $\mathcal{X}$-cellular morphism. Hence each morphism in $\mathcal{K}$ is a retract of a $F(\mathcal{X})$-cellular morphism. Thus $(\mathcal{K}, \mathrm{Iso})$ is cofibrantly generated. □

Given a complete and cocomplete category $\mathcal{K}$, the choice $\mathcal{C} = \mathcal{K}$ and $\mathcal{W} = \mathrm{Iso}$ yields a model category structure on $\mathcal{K}$. The corresponding two weak factorization systems are $(\mathcal{K}, \mathrm{Iso})$ and $(\mathrm{Iso}, \mathcal{K})$ and the homotopy category $\mathrm{Ho}(\mathcal{K}) = \mathcal{K}$. We will call this model category structure *trivial*.

Corollary 3. *Let $\mathcal{K}$ be a complete, cocomplete and bounded category. Then the trivial model category structure on $\mathcal{K}$ is cofibrantly generated.*

Proof. Following Proposition 2, it suffices to add that the weak factorization system $(\mathrm{Iso}, \mathcal{K})$ is cofibrantly generated by $\mathcal{X} = \{\mathrm{id}_O\}$ where O is an initial object of $\mathcal{K}$. □

Theorem 4. *Vopěnka's principle is equivalent to the fact that every cofibrantly generated model category is Quillen equivalent to a combinatorial model category.*

Proof. Necessity follows from [6]. Under the negation of Vopěnka's principle, [2], 6.12 presents a complete bounded category $\mathcal{A}$ with the following properties

(1) For each regular cardinal λ, there is a λ-filtered diagram $D_\lambda \colon \mathcal{D}_\lambda \to \mathcal{K}$ whose only compatible cocones δ_λ are trivial ones with the codomain 1 (= a terminal object),

(2) For each λ, id_1 does not factorize through any component of δ_λ.

Since, following (1), δ_λ is a colimit cocone for each λ, (2) implies that 1 is not λ-presentable for any regular λ. Condition (2) is not stated explicitly in [2] but it follows from the fact that there is no morphism from 1 to a non-terminal object of $\mathcal{A}$. In fact, $\mathcal{A}$ is the full subcategory of the category $\mathbf{Gra}$ consisting of graphs A without any morphism $B_i \to A$ where B_i is the rigid class of graphs indexed by ordinals (whose existence is guaranteed by the negation of Vopěnka's principle). The existence of a morphism $1 \to A$ means the presence of a loop in A and, consequently, the existence of a constant morphism $B_i \to A$ (having a loop as its value).

Assume that the trivial model category $\mathcal{A}$ is Quillen equivalent to a combinatorial model category $\mathcal{M}$. Since $\mathrm{Ho}\,\mathcal{M}$ is equivalent to $\mathcal{A}$, it shares properties (1) and (2). Moreover, since $\mathrm{Ho}\,\mathcal{K} = \mathcal{K}$, the diagrams D_λ are diagrams in $\mathcal{K}$. It follows from the definition of Quillen equivalence that the corresponding diagrams in $\mathrm{Ho}\,\mathcal{M}$ (we will denote them by D_λ as well) can be rectified. It means that there are diagrams $\overline{D}_\lambda$ in $\mathcal{M}$ such that $D_\lambda = P\overline{D}_\lambda$; here, $P\colon \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ is the canonical functor. Following [3] and [8], there is a regular cardinal λ_0 such that the replacement functor $R\colon \mathcal{M} \to \mathcal{M}$ preserves λ_0-filtered colimits. R sends each object M to a fibrant and cofibrant object and the canonical functor P can be taken as the composition QR where Q is the quotient functor identifying homotopy equivalent morphisms.

Let

$$(\overline{\delta}_{\lambda d}\colon \overline{D}_\lambda d \to M_\lambda)_{d\in\mathcal{D}_\lambda}$$

be colimit cocones. Then

$$(R\overline{\delta}_{\lambda d}\colon R\overline{D}_\lambda d \to RM_\lambda)_{d\in\mathcal{D}_\lambda}$$

are colimit cocones for each $\lambda > \lambda_0$. Following (1), $RM_\lambda \cong 1$ for each $\lambda > \lambda_0$. The object RM_{λ_0} is μ-presentable in $\mathcal{M}$ for some regular cardinal $\lambda_0 < \mu$. Since RM_{λ_0} and RM_μ are homotopy equivalent, there is a morphism $f\colon RM_{\lambda_0} \to RM_\mu$. Since f factorizes through some $R\overline{\delta}_{\mu d}$, id_1 factorizes through some component of δ_μ, which contradicts (2). □

While the weak factorization system $(\mathrm{Iso}, \mathcal{K})$ is cofibrantly generated in the sense of [5], it is not true for $(\mathcal{K}, \mathrm{Iso})$ because the complete, cocomplete and bounded category in [2], 6.12 is not locally presentable just because it contains a non-presentable object. Thus we do not know whether Vopěnka's principle follows from the original result from [6].

The proof above does not exclude that $\mathcal{A}$ *has a combinatorial model*, i.e., that there is a combinatorial model category $\mathcal{M}$ such that $\mathcal{A}$ is equivalent to $\operatorname{Ho}\mathcal{M}$.

Proposition 5. *Assume the existence of a proper class of compact cardinals and let $\mathcal{K}$ be a complete, cocomplete and bounded category. Then the trivial model category $\mathcal{K}$ has a combinatorial model if and only if $\mathcal{K}$ is locally presentable.*

Proof. If $\mathcal{K}$ is locally presentable the trivial model category $\mathcal{K}$ is combinatorial. Assume that the trivial model category $\mathcal{K}$ is equivalent to $\operatorname{Ho}\mathcal{M}$ where $\mathcal{M}$ is a combinatorial model category. Let $\mathcal{X}$ be a dense subcategory of $\mathcal{K}$. Following [8], 4.1, there is a regular cardinal λ such that

(1) $\mathcal{X} \subseteq P(\mathcal{M}_\lambda)$ where $\mathcal{M}_\lambda$ denotes the full subcategory of $\mathcal{M}$ consisting of λ-presentable objects,
(2) The composition $H = E_{P(\mathcal{M}_\lambda)} \cdot P$ preserves λ-filtered colimits.

Since $P(\mathcal{M}_\lambda)$ is dense in $\mathcal{K}$ (see Lemma 1), $E_{P(\mathcal{M}_\lambda)}$ is a full embedding. Hence $\mathcal{K}$ is the full image of the functor H, i.e., the full subcategory on objects $H(M)$ with M in $\mathcal{M}$. Following [7], Corollary of Theorem 2, $\mathcal{K}$ is locally presentable. $\square$

Vopěnka's principle is stronger than the existence of a proper class of compact cardinals. Thus, assuming the negation of Vopěnka's principle but the existence of a proper class of compact cardinals, there is a cofibrantly generated model category without a combinatorial model.

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