ROOTS AND LOCAL COLIMITS

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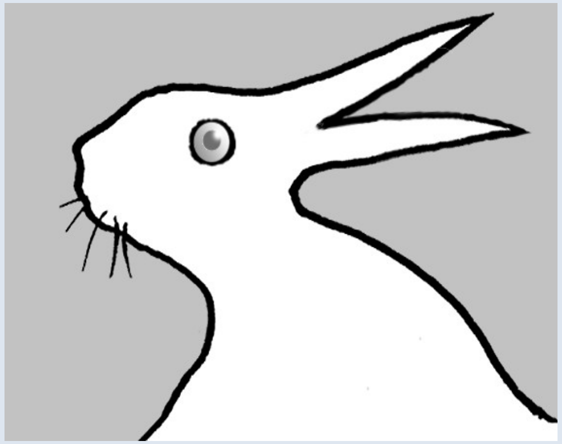
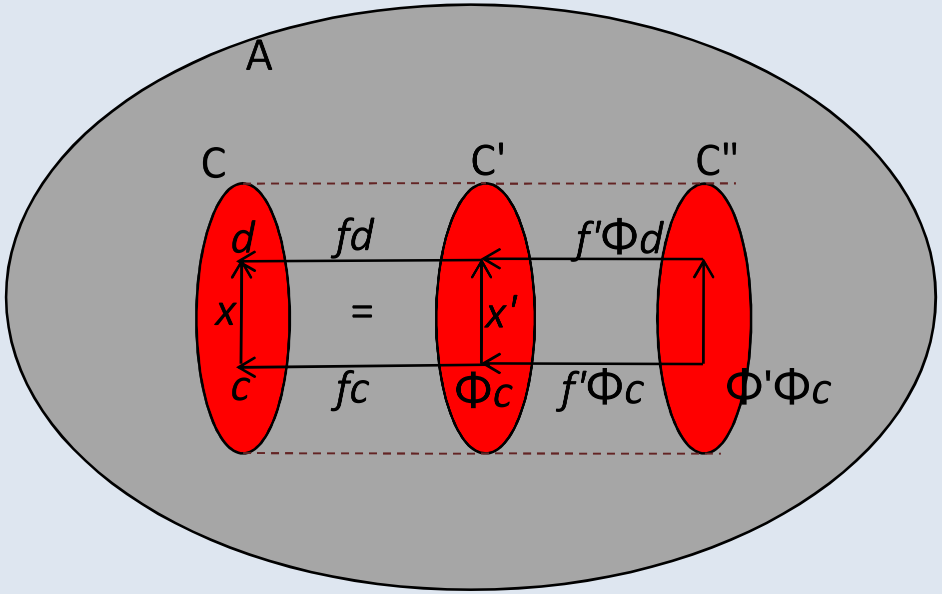


COLIMIT ---------> MULTI-COLIMIT Unicity

LOCAL COLIMIT Unicity

LOCALLY FREE COLIMIT DIAGRAM Non-unicity

CYLINDERS OF A CATEGORY



Let C and C ′ be two sub-categories of a category A. An A-*cylinder* from C to C' is a pair (Φ, f) of a map Φ from |C| to |C'| and a map *f* sending each object *c* of C on a morphism *fc* from Φ(*c*) to *c* such that, for each $x : c \rightarrow d$, there exists at least one x' making the square above commutative.

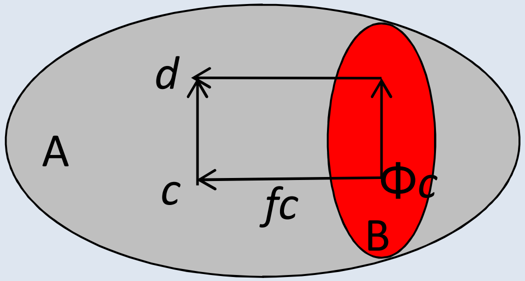
The A-cylinders are the morphisms of the category A-**Cyl** whose objects are the sub-categories of A. An inversible (Φ, *f*) is such that Φ is a 1-1 correspondence and *fc* an isomorphisme for each *c* ; then Φ extends in an isomorphism from C onto C'.

If A is a poset, an A-cylinder from C to C' reduces to a contraction R from C to C' (that is, R$c < c$ for each c).

COREFRACTS AND ROOTS

A *corefract* of a category A is a subcategory B such that there exists an A-cylinder **(**Φ**,** *f*) from A to B with *fb* = *b* for each *b* in |B|. It is equivalent to say that B is a full sub-category of A which is weakly coreflective.

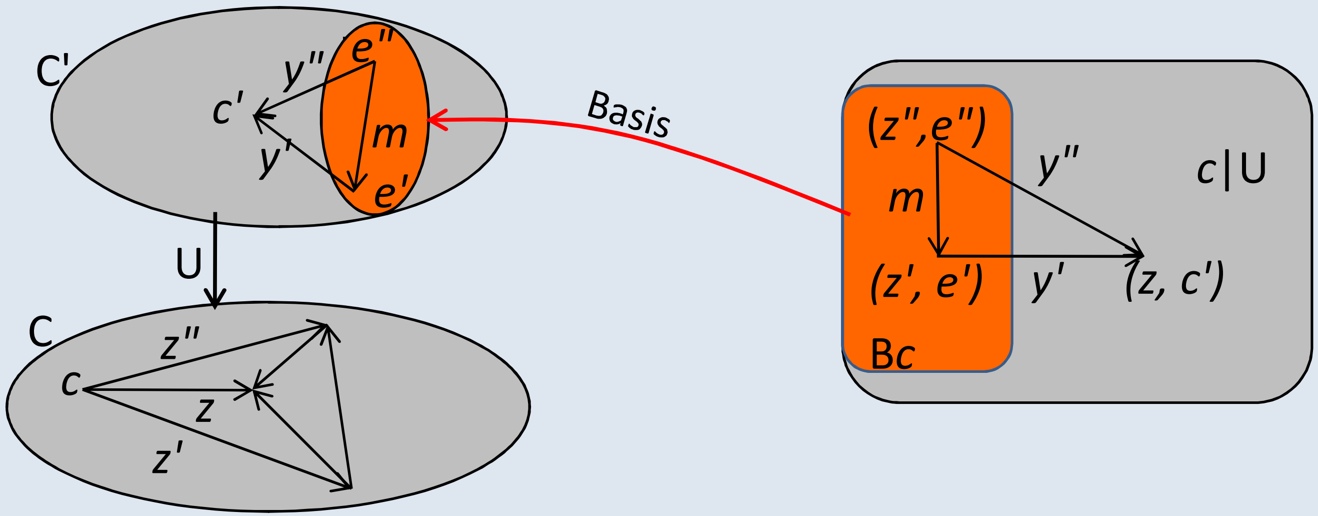
A category B is called a *root* if each B-cylinder from B to B is invertible. It is a *root of the category* A if B is a root, a full sub-category of A and if there exists at least one A-cylinder (Φ, *f*) from A to B. In this case, B is a minimal corefract of A and B has no proper corefract.



> **Theorem.** ***A sub-category* B *of* A *is a root of* A *iff* B *is a corefract of* A *and is a root. If such a* B *exists, it is unique up to an isomorphism, both in* Cat *and in* A-Cyl .**

Examples. 1. A groupoid is a root iff it is a sum of groups.
2. A finite poset is a root iff no element has a predecessor. A root of a poset A corresponds to a minimal full coreflective sub-category.

ADJOINT-ROOT FUNCTOR



Let U be a functor from C' to C. If *c* is an object of C, a root Bc of the comma-category *c*|U is called a U-*universal root generated by c.* It is unique up to an isomorphism.

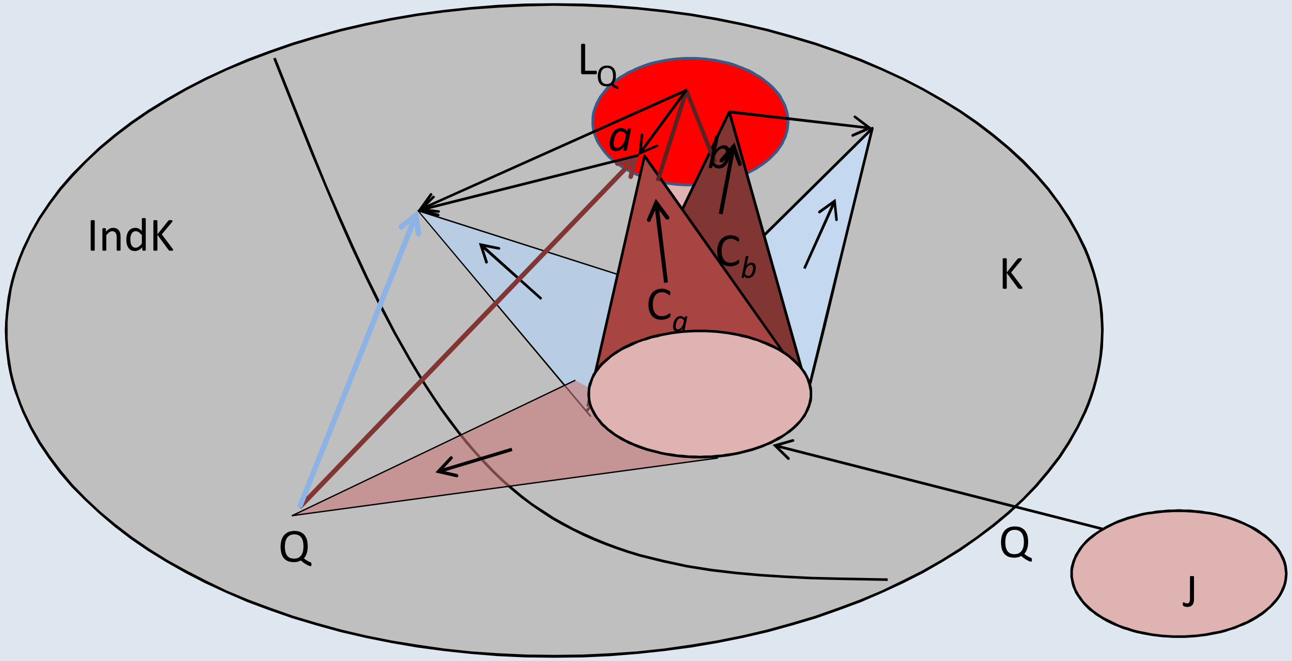
Examples. If B is a finite poset, it is a U-universal root generated by *c* iff B is a leasr full coreflective subcategory of *c*|U. If B is a groupoid, it is a U-universal root generated by *c* iff B is a sum of groups and a full initial subcategory of *c*|U.

Theorem. *If c generates a* U-*universal root* Bc, *the restriction β*c *of the basis functor to* Bc *is a* Pro-U *universal object generated by c. If* Bc *exists for each object c of* C, *the map sending c to β*c *extends into a functor from* C *to* ProC', *called an adjoint-root functor of* U.

(This theorem generalizes a result of Guitart for locally free diagrams.)

LOCAL COLIMITS

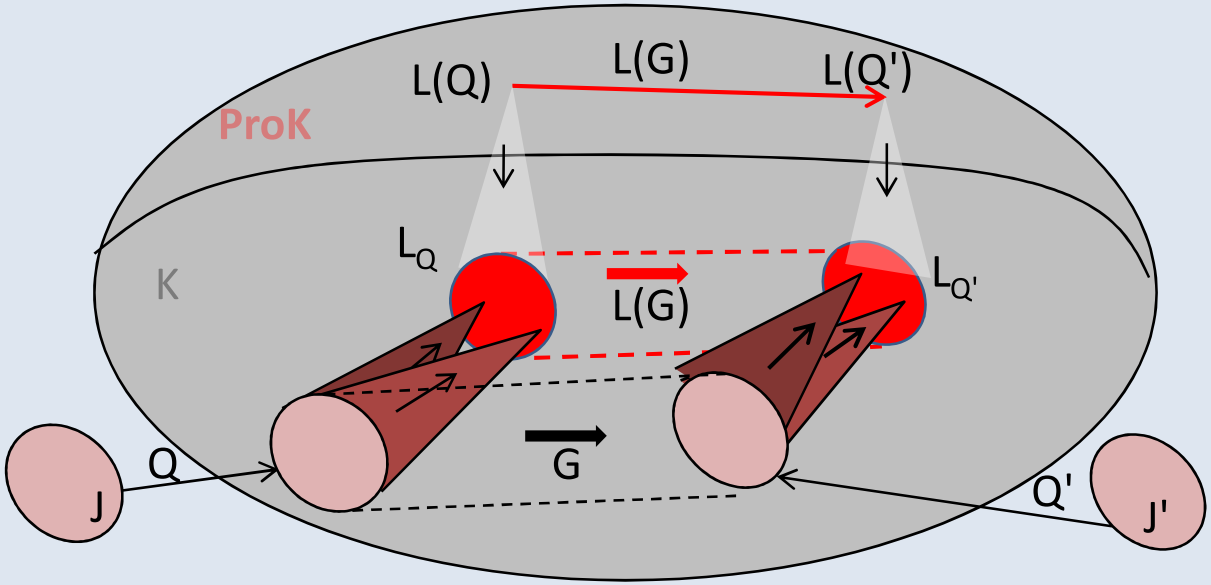
Let U be the 'insertion' of the category K into the category of its ind-objects IndK. If Q is a functor from J to K, the category Q|U 'is' the category CQ of cones with basis Q. A *colimit-root of* Q is a root B_Q of CQ to which the restriction of the 'vertex' functor V is 1-1; then the image L_Q of B_Q by V is called *a local colimit* of Q.



***If it exists, the local colimit L_Q is unique up to an isomorphism of* Cat.**

L_Q is a local colimit of Q iff L_Q is a root and if each object *a* of L_Q is the vertex of a cone C_a with basis Q, these cones satisfying: (i) morphisms of L_Q commute with these cones; (ii) each cone with basis Q has a weakly terminal factorization through one of them.

COLIMIT-ROOT FUNCTOR



Theorem. ***There is a 'colimit-root functor'* L *from the full sub-category of* IndK *whose objets* Q *admit a local colimit* L_Q *to* ProK *sending* Q *to the insertion* L(Q) *of* L_Q *into* K.**

L_Q is reduced to one object cQ iff Q admits a colimit cQ. If L_Q is discrete, a local colimit reduces to a *multi-colimit* in the sense of Diers (1971). Local colimits are particular locally free colimit diagrams in the sense of Guitart & Lair (1980).

Q has a colimit-root B_Q which is a groupoid iff B_Q is a sum of groups which is a full and initial sub-category of the category of cones CQ. If K is an accessible category which has finite products, every functor to K has such a colimit-root (Lair, 1983).

APPLICATIONS

1. Galois catégories

By definition, a *Galois category* is a category equivalent to the category of continuous actions of a profinite group on a finite discrete space. We have:

A Galois category admits a root and is equivalent to a root.

Let C be the category whose objects are pairs (H, P) of a commutative field H and a polynomial, C' (resp. C") the full sub-category whose objects are the pairs (H, P) where P has all its roots (resp. has at least one root) in H. If U' and U" are the insertions of C' and C" into C, then

Each* (H, P) *generates a* U'*-universal root which is a group. If* P *is separable,
(H, P) generates a* U"*-universal root which is a Galois category.

2. Roots in Directed Algebraic Topology

In his characterization of the fundamental category in directed algebraic topology, Marco Grandis obtains different examples of categories admitting roots and coroots, and also of categories with no root nor coroot. In particular he proves that

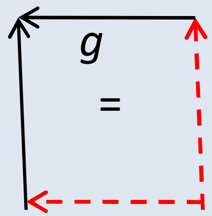
***The past spectrum* $Sp^{--}X$ *of a category* X *with no past*$^{-}$ *branchings is a root of* X.**

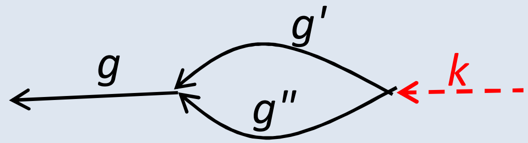
The *past spectrum* is a full coreflective sub-category of X with just one object in every past regularity class, defined as follows.

Two objects are *past regularity equivalent* if there is a past regular morphism *g* between them, meaning that :

(i) for any morphism with the same codomain as *g*, there is a commutative square joining them;

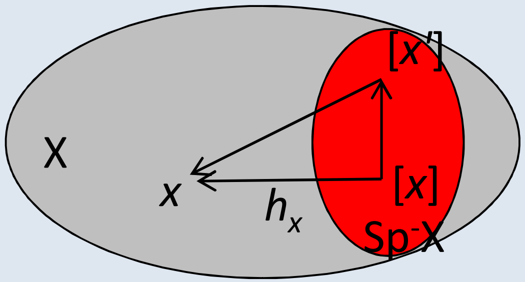
(ii) if *gg' = gg'',* then there exists a *k* such that *g'k = g''k*.



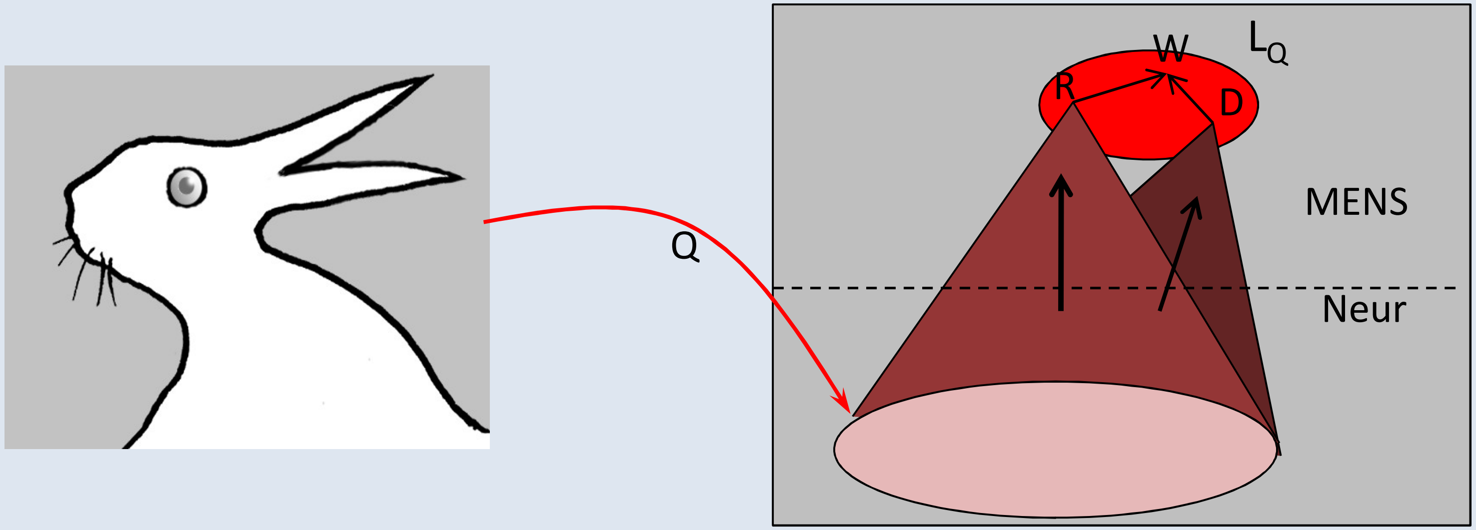


An object *x* is *past-branching* if there exists a map *g* with codomain *x* which does not satisfy (i).

The *past spectrum* is a full sub-category of X such that: (i) it has just one object [*x*] in every past regularity class x^- and there is just one morphism h_x from [*x*] to *x;* (ii) every morphism from [*x'*] to *x* factors uniquely through h_x. It is the least full coreflective subcategory. Its objects are the past branching points and a minimal object.



MENS MOTIVATION



Local colimits were introduced to model ambiguous figures in the model **MENS** for a cognitive and mental system which we develop with Jean-Paul Vanbremeersch..

MENS is constructed by complexification processes, from the evolutive system of neurons. In particular, a mental image is modeled by the colimit of a diagram Q in Neur modeling a synchronous assembly of visual neurons Q which the image activates.

For an ambiguous object such as the duck-rabbit, the colimit is replaced by a local colimit L_Q of Q : it admits 3 objects R, D, W respectively modeling the images of the 2 separate objects (Rabbit and Duck) and the Whole image W, and there are 2 arrows from R and D to W.