From Analysis to Living Systems through Category land

by

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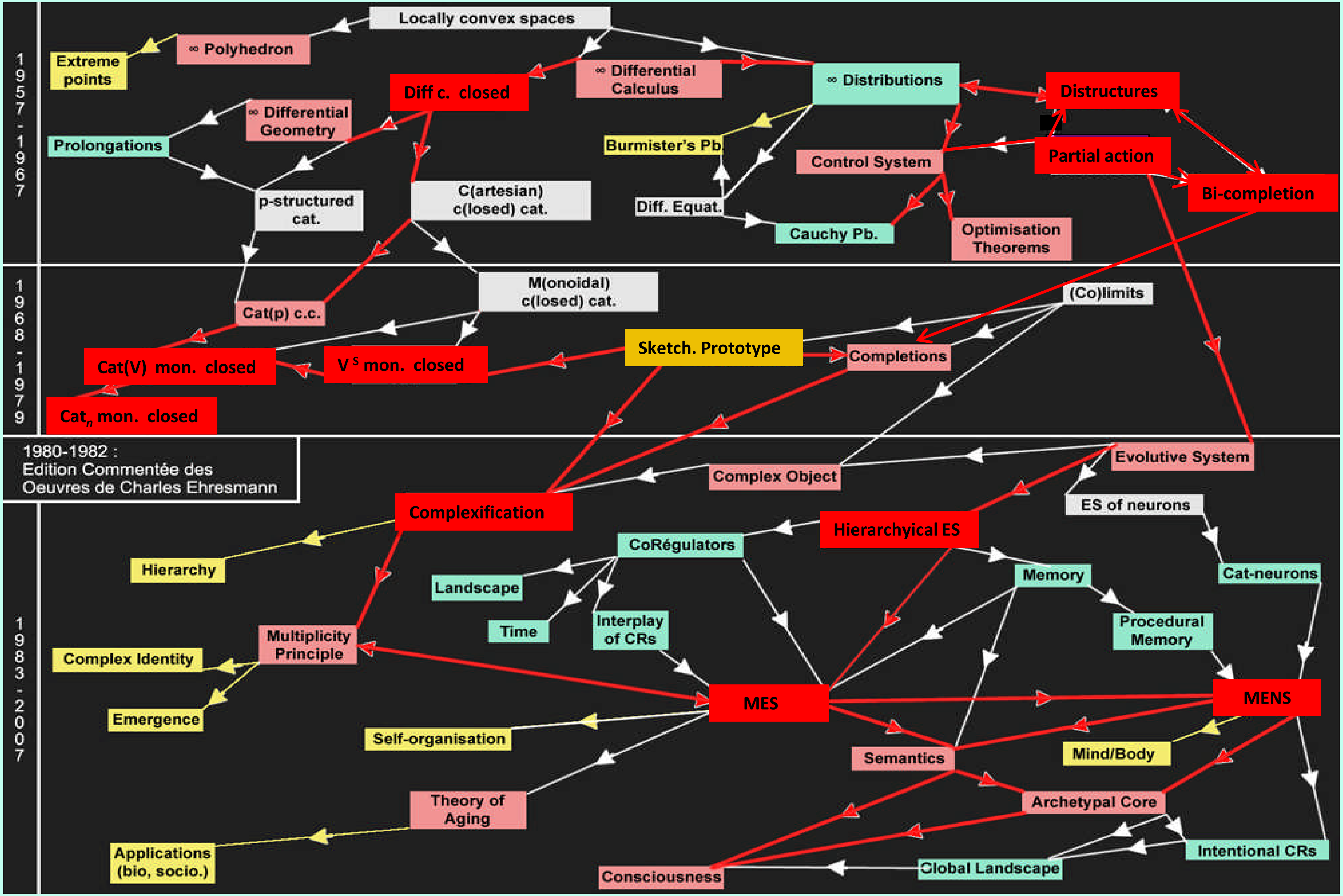
INITIALLY AN ANALYST

In 1956, first research under the direction of Choquet on his now well known Theorem on extreme points of convex sets. Long discussions with him made me familiar with the intricate properties of infinite dimensional locally convex spaces.

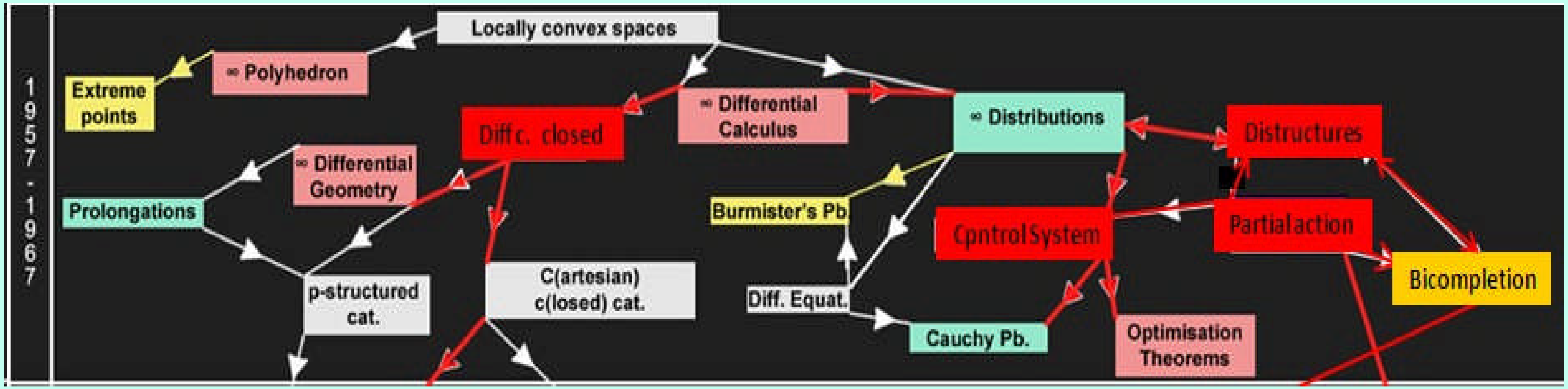
These properties are used in my 3rd cycle thesis which deals with convex geometry and applications to optimization, extending results of Rosenbloom . The main result is a new definition of a polyhedron which extends to the infinite dimensional case, namely as a convex set such that each supporting cone is closed.

This photo shows Charles and Choquet at the "Congrès des Mathématiciens d'expression latine" (Nice, 1957). Our 22 years long close relation with Charles began at this time.

DIAGRAM OF MY RESEARCH WORK

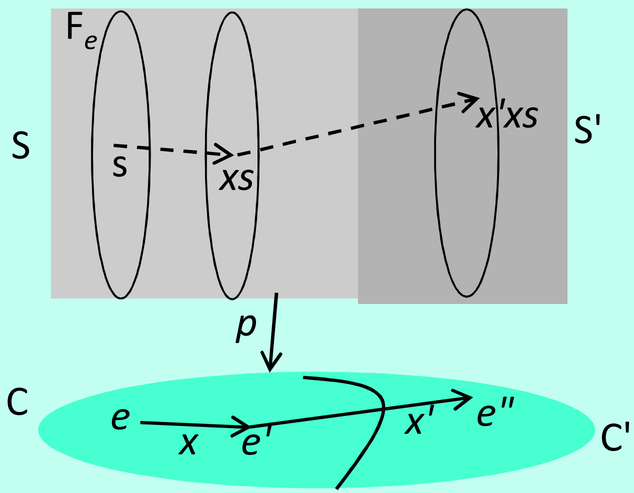


FROM ANALYSIS
TO
CATEGORY THEORY



DISCOVERY OF CATEGORY LAND

I discovered categories through Charles' paper "Gattungen von lokalen Struckturen" (1957), the first where he uses the word 'category'. In it he introduced the action of a (local) category C on a (local) set S, calling S a (local) *species of structures* over C.



He proves that it corresponds to a functor F from C to Set, such that the 'fibers' F_e are disjoint, and S is their union.
He also defines and characterizes the associated discrete fibration *p* (called 'category of hypermorphisms') on the set C∗S of 'composable' pairs (*s, x*).

The main theorem is the *"Complete Enlargement Theorem*" which translates in this categorical setting the construction of the species of local structures associated to a pseudogroup of transformations. It comes in 2 parts:

(i) If a subcategory C of C' acts on S, extension of this action in an action of C' on a set S';

(ii) If C and C' are "local categories" (in modern terms special categories internal to the category of locales) and S a locale, extension of a "local" action into a complete local action of C' on a locale S' (complete meaning that it satisfies a "sheaf" property).

Applications are given to the construction of locally trivial fibre bundles and foliations.

DISTRIBUTIONS AND PARTIAL ACTIONS

Motivated by problems in functional analysis and differential equations, I developed a categorical frame, the *theory of distructures* (translated in modern terms in my Calais presentation, 2008) for studying different kinds of "generalized functions", and in particular for extending Schwartz distributions to infinite dimensional locally convex vector spaces (lcs).

> **" Une distribution est, localement, une dérivée d'une fonction continue "**
>
> Laurent Schwartz, *Théorie des distributions* (1950), p. 8

Imitating Charles' Complete Enlargement Theorem, the above citation suggested to construct distributions on a lcs E, with values in an lcs E' through a "bi-completion" process:

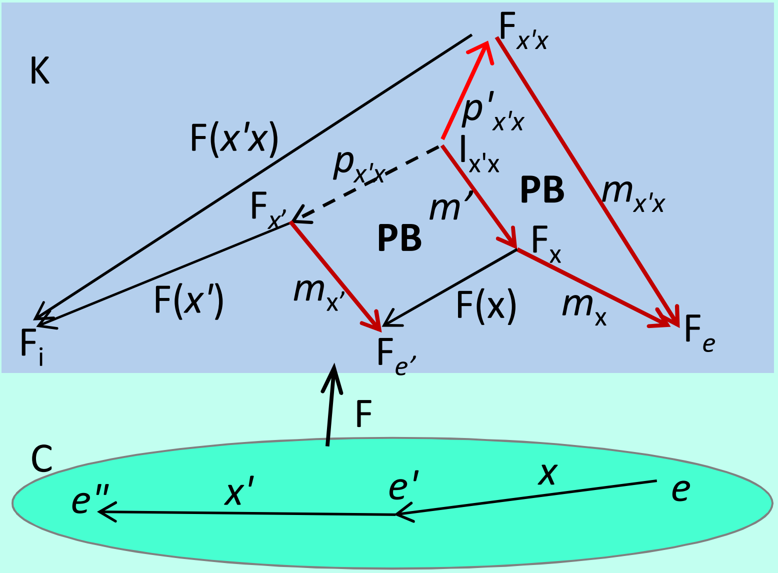
(i) Differential operators *d* define a 'partial action' on the lcs of continuous maps from an open subset U of E to E', the composite *df* being defined only iff *f* has a *d*-derivative. This partial action is extended into an action (enriched in Lcs), leading to the presheaf of finite order E'-valued distributions over E.

(ii) A sheafification of this presheaf leads to the sheaf of E'-valued distributions over E.

It led to the notion of a *partial action* of a category C on a set S, and to its extension to an action of C on S'. A *partial action* is defined as an action, except that, for x: $e \to e'$ the 'composition' F(x): $s \mapsto xs$ is only defined on a *subset* F_x of the fiber F_e , and the 'transitivity' condition is: the composite *x'(xs)* is defined if and only if *xs* and *(x'x)s* are both defined.

This notion is 'enriched' into that of a (K, M)-semi-functor for defining distructures, and, later, 'internalized' for studying control problems.

ENRICHED PARTIAL ACTION. DISTRUCTURES



Let K be a category, M a class of monomorphisms containing |K|, stable by pullbacks and with at most one *m* between 2 objects. A (K, M)-*semi-functor* on C is a map from C to K such that:

(i) $F(e)$ is an object F_e for e in |C|.

(ii) For $x: e' \to e$, $F(x): F_x \to F_{e'}$ where F_x is an object such that there exists $m_x: F_x \to F_e$ in M.

(iii) *Transitivity condition*: for $x': e'' \to e'$ in C, the pullback $I_{x'x}$ of $(m_{x'}, F(x))$ is also the pullback of $(m_x, m_{x'x})$ and $F(x'x)p_{x'x} = F(x')p'_{x'x}$ where $p_{x'x}$ and $p'_{x'x}$ are projections of the PB's.

Associated Functor Theorem. If K admits pullbacks and 'enough' colimits, there is a reflexion P from the category SS(K, M) of (K, M)-semi-functors to the category Diag(K) of functors to K.

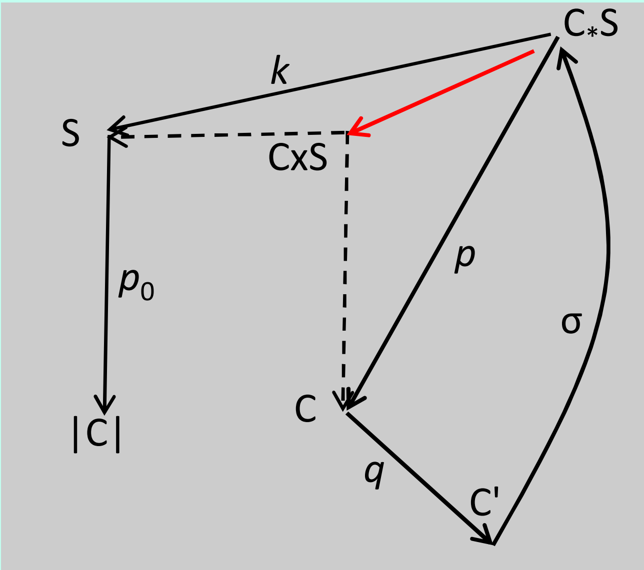
A *generator of* (K,M)-*distructures* on a category H is a functor B from H^{op} to the category SS(K M). Its composite B': $H^{op} \to$ Diag(K) with the reflection P: SS(K, M) $\to$ Diag(K) is the *presheaf of* K-*distructures generated by* B. If H is equipped with a Grothendieck topology (or a convenient order), the sheaf associated to B' is the *sheaf of* (K, M)-*distructures.*

In my thesis K is the category Lcs of locally convex vector spaces and M the insertions into a lcs of a vector subspace with a finer topology. In particular I give an explicit construction of the *sheaf of vector distributions on infinite dimensional manifolds*.

DIFFERENTIABILITY. CONTROL SYSTEMS

To define these infinite dimensional ditributions, I had to develop a 'good' differential calculus in lcs (1962). In 1964, trying to extend the main theorems of differential geometry (e.g., the *transitivity of prolongations* for manifolds) in this frame, I realized that the category of differentiable maps Diff had to be 'Cartesian closed'. At this end, I replaced topologies by quasi-topologies (or "Limesräum"); this idea was later taken back by Keller, Frölicher and others.

'Internal' partial actions are at the root of my work on Control Systems, giving a categorical frame to model Cauchy boundary problems and variation problems (1963-1966). The main results give *optimisation theorems* for solutions of a control system in terms of infinite dimensional vector distributions; they specify and extend the dynamic programing method of Bellman.



A *nucleus of action* is a partial action of a topological category C on the topological space S such that the composition law *k*: (x, s) |→ xs be continuous from an open subset C_*S of CxS to S. It has an associated 'partial fibration' *p*: C_*S → *C*..

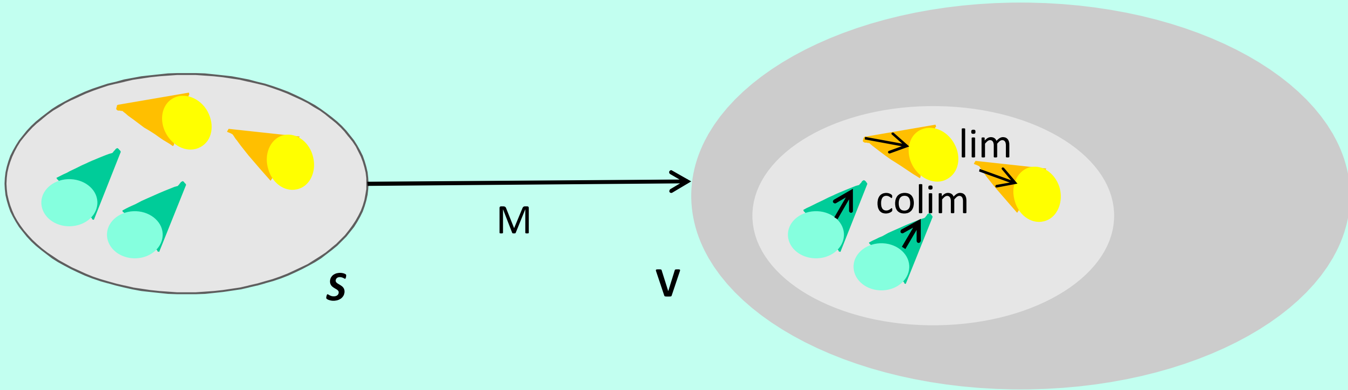
A *control system* is a nucleus of action with a topological functor *q*: C → C'. A *solution* of this system is a topological functor σ: C' → C_*S section of *qp*. *Differentiable Control Systems* are obtained by replacing Top by Diff.

ON SKETCHES AND THEIR MODELS

2nd Conference on the Algebra of Categories (AMIENS 1975); photo taken in Chantilly

The notions developed for adapting categorical notions to analysis paved the way for our research with Charles. From 1968 up to Charles' death in 1979, our joint papers had to do with problems about sketches and the categories of their models. These problems were also at the basis of the work of several members of our research team "Théorie et Applications des Catégories" (Paris-Amiens) which we developed during this period.

SKETCHES AND THEIR MODELS



I and ***J*** being classes of small categories, an (***I, J***)-*sketch* ***S*** is a (neo)category ∑ equipped with a set of projective cones indexed by ***I*** and a set of inductive cones indexed by ***J***.
A *prototype* is a sketch in which the projective (resp. inductive) cones are limit (resp. colimit) cones.
An (***I, J***)-*type* is a prototype in which each functor indexed by an element of ***I*** has a limit (resp. (resp. of ***J*** has a colimit).

Theorem (1972)**.** The 2-category of (***I, J***)-sketches admits a 2-reflection into the 2-category of (***I, J***)-prototypes and into that of (***I, J***)-types; the prototype and the (***I, J***)-type so associated to a sketch are explicitly constructed (by induction) up to an *isomorphism*.

A *model* M *of* the sketch ***S*** into a category **V** is a functor from ∑ to **V** sending the projective cones on limit-cones and the inductive cones on colimit cones; it also defines a model of the prototype associated to ***S*.** The category $\mathbf{V}^{S}$ of models of ***S*** in **V** is a full sub-category of the category $\mathbf{V}^{\Sigma}$**.**

MONOIDAL CLOSED CATEGORIES OF MODELS

In the second part of our 1972 paper, we give conditions for the category of models $\mathbf{V}^{S}$ of a projective sketch ***S*** in a symmetric monoidal closed category **V** to be monoidal closed.

S being a projective sketch, we say that ***S*** is *cartesian* if Set^{S} is cartesian closed. We give different equivalent conditions (and later Street has given still other ones).

Theorem. We suppose that ***S*** is cartesian, and that **V** admits ∑-ends and coproducts indexed by the Hom of ∑. If the tensor product commutes with ***I***-projective limits and if the categories in **I** are connected, then the category $\mathbf{V}^{S}$ has a symmetric monoidal closed structure, which is a sub-structure of that of the category $\mathbf{V}^{\Sigma}$. In particular if **V** is cartesian closed, then so are $\mathbf{V}^{S}$ and Cat(**V**).

For **V**-internal categories, we have a stronger result and give a construction of the internal Hom generalizing that of natural transformations from C to C' as functors from C to the category *Sq*C' of commutative squares of C'.

Theorem. Let **V** be a symmetric monoidal closed category which admits pullbacks and equalisers of pairs. If its tensor product commutes with pullbacks, then Cat(**V**) has a symmetric monoidal closed structure whose internal **Hom** is given by

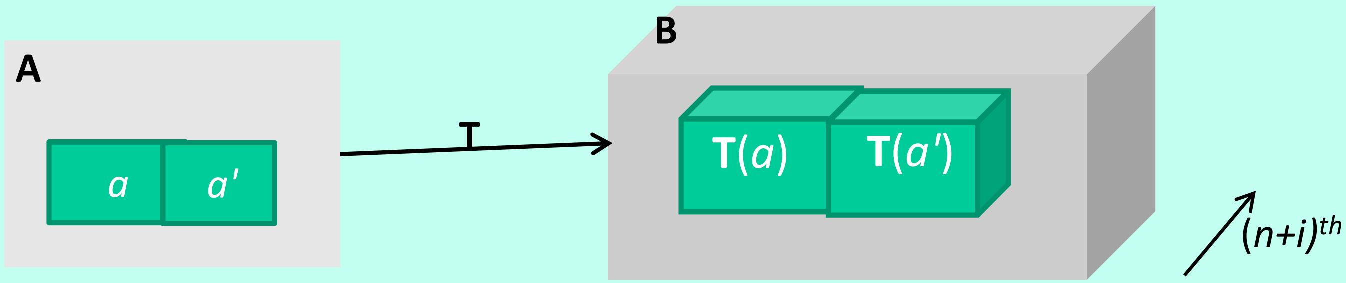
$$\mathbf{Hom}(C, C') = \int \mathbf{V}(C(-), SqC'(-))$$

where *Sq*C' is the **V-**internal category of commutative squares of C'.

THE CATEGORY MCat OF MULTIPLE CATEGORIES

My last series of papers with Charles (1974-1979) studies the categories Cat_n of n-fold categories. An n-fold category **A** is defined by induction as a category internal to Cat_{n-1}; it "is" a class A of "blocks", equipped with n laws of categories 2x2 permutable. Given a sequence $i,\dots j$ of m laws of **A**, we denote by $\mathbf{A}^{i,\dots,j}$ the m-fold category on A obtained by retaining only these laws.

Let MCat be the category whose objects are all the n-fold categories for each n, and the morphisms the *multiple* functors defined as follows: there is a multiple functor **T** from the n-fold category **A** to the p-fold category **B** iff $n \leq p$ of **B**, and then **T** is an n-fold functor from **A** to $\mathbf{B}^{1,\dots,n}$. MCat contains all the categories Cat_n for $n \geq 1$ as full sub-categories.

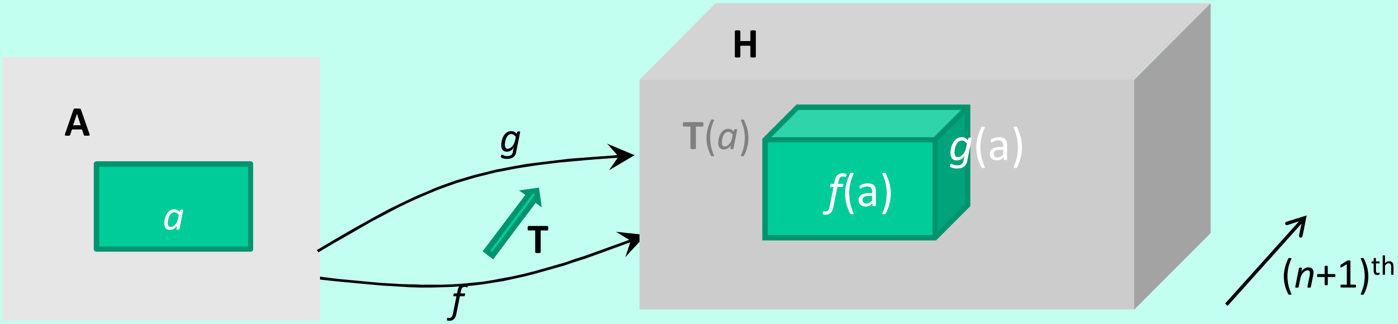


Theorem. MCat has a "partial" monoidal closed structure:

1. The internal Hom, **MCat**(**A, B**), is defined only if the multiplicity n of **A** is less than the multiplicity p of **B**, and then it is the $(p\text{-}n)$-fold category of multiple functors **T** from **A** to **B,** its compositions being deduced from the p-n last compositions of **B.**
2. The tensor product of (A, B) is the $n+p$ fold category **A ◊ B** on AxB whose i-th category is

$$A^{\mathrm{dis}} \times B^{i} \quad \text{if} \quad i \leq p \qquad \text{and} \qquad A^{i-p} \times B^{\mathrm{dis}} \quad \text{if} \quad p < i \leq n+p.$$

LIMITS RELATIVE TO A MULTIPLE CATEGORY



Let **H** be an (n+1)-fold category. If **A** is an n-fold category, **MCat**(**A, H**) is a category whose objects are the n-fold functors f from **A** to the n-fold subcategory of **H** consisting of its objects for the (n+1)-th law. Then a multiple functor **T**: **A** → **H** can be considered as a **H**-*transformation from f to g*.

Let |**H**| be the sub-category of H^{n+1} formed by the blocks which are objects for the n first laws. A cofree object generated by f with respect to the functor "insertion" of |**H**| into **MCat**(**A, H**) is called a **H**-*wise limit of f.*

We say that **H** is *representable* if $H^{1,\ldots,n}$ admits **H**-wise limits indexed by $2^{¤n}$, where the n-fold category $2^{¤n}$ is defined by induction by: $2^{¤1}$ = 2 and $2^{¤(n+1)}$ = $2^{¤n}$¤2.

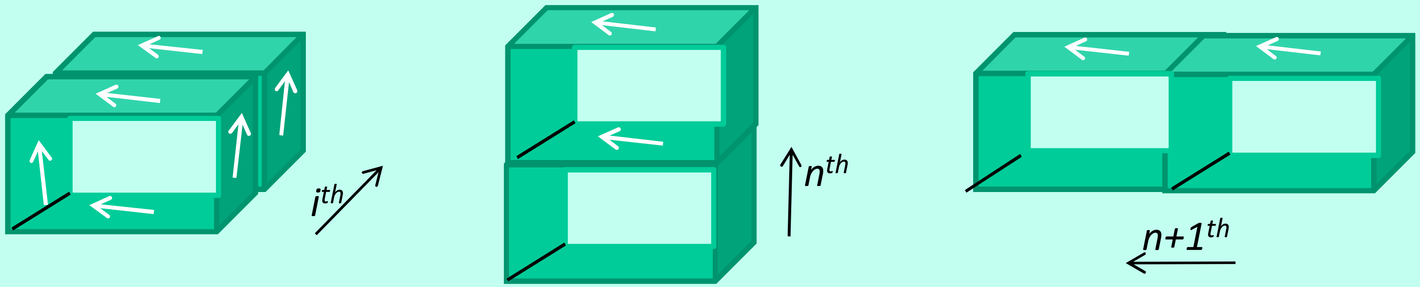
Theorem. 1. The functor -¤$2^{¤p}$: Cat_n → Cat_{p+n} is left adjoint to the functor from Cat_{p+n} to Cat_n 'omitting' the n first compositions; and the category Cat_n is the inductive closure of {$2^{¤n}$}.

2. If **H** is representable, so are ^{1}*Sq***H** and *Cub***H**.

3. If **H** is representable and |**H**| complete, then $H^{1,\ldots,n}$ is **H**-wise complete.

For double categories, these results have been extended by Grandis & Paré (1998) to double functors.

THE FUNCTORS Square AND Link ON Cat$_n$

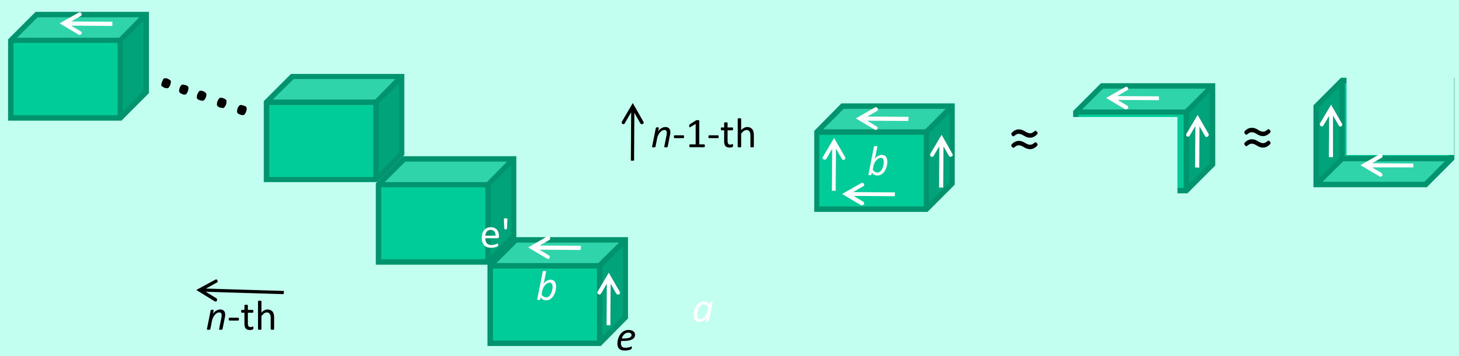


There is a functor ^1Sq: MCat → MCat mapping an n-fold category **A** on the n+1-fold category ^1Sq**A** on the class of commutative squares of A^1: its n-1 first laws are that of MCat(2x2, **A)**, the n-th is the vertical composition of squares and the (n+1)-th their horizontal composition.

Theorem. The functor from Cat$_n$ to Cat$_{n+1}$ restriction of ^1Sq has a left adjoint *Link* such that *Link*(^1Sq**A**) is isomorphic to **A** for each n-fold category **A.**

The n-fold category *Link***B** is constructed as a quotient of the category of all the paths on the graph G whose vertices *e* are objects for the last 2 laws of **B**, the edges from *e* to *e'* being all the blocks *b* admitting *e* and *e'* for "extremities", the relation identifying *b* with the paths below; so *Link***B** is formed of classes of srings of objects for B^n and B^{n+1}.

For a 2-category **C**, *Link***C** is isomorphic to the category of components of C^1.



MONOIDAL CLOSED STRUCTURES ON $\mathbf{Cat}_n$

Theorem. Cat_n is cartesian closed, with internal Hom:

$$\mathbf{Hom}_n(\mathbf{A}, \mathbf{B}) = \mathbf{MCat}(\mathbf{A}, {}^1Sq^n\mathbf{B})$$

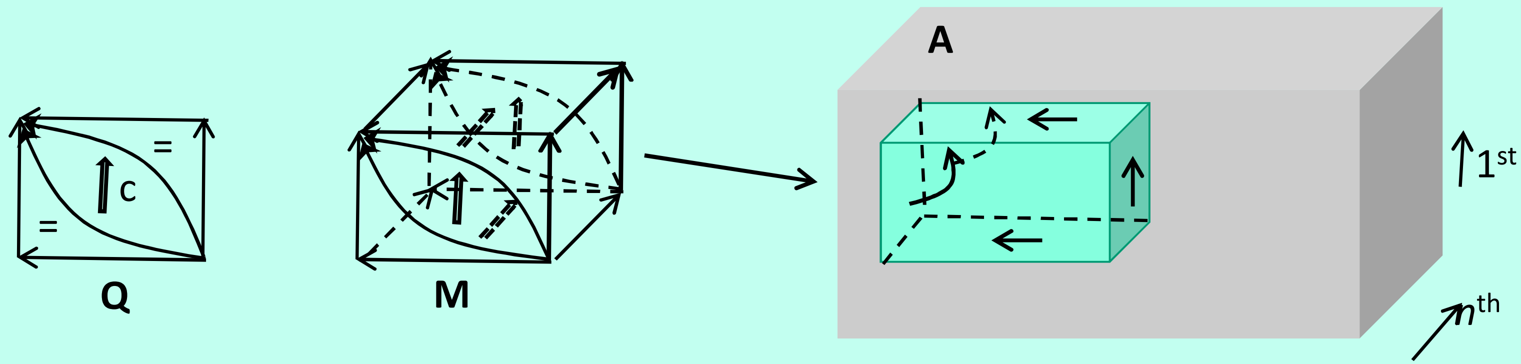
where ${}^1Sq^n$ is the n-th composite of the functor ${}^1Sq$.

The proof uses the fact that

$$\mathbf{A} \times \mathbf{B} = Link^n(\mathbf{A} \lozenge \mathbf{B})^{1,n+1,\ldots,i,\, i+1,\ldots,\, n,2n}$$

Other monoidal closed structures can be defined on Cat_n by 'laxifying' the preceding construction, replacing the functor *Square* by the functor *Cube* defined as follows:

Let **Q** be the double category with one 2-cell c and four 1-cells forming a square "commutative up to c ", and let $\mathbf{M} = \mathbf{Q} \times \mathbf{2}$, where **2** is the double category on 2 whose 2nd law is discrete. If **A** is an n-fold category, a *cube* of **A** is a double functor from **M** to $A^{n,1}$. The $(n+1)$fold category *Cub***A** has for blocks the cubes of **A**, the laws being defined as for squares.



THE INTERNAL HOM LaxHom_n ON Cat_n

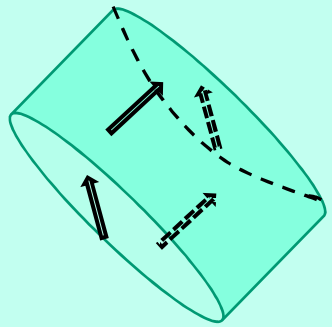
Theorem. The functor *Cub*: $\text{Cat}_n \to \text{Cat}_{n+1}$ mapping **A** on *Cub***A** has a left adjoint *LaxLink*.

The construction of *LaxLink***B** 'laxifies' that of Link using the relation

so that its blocks are classes of strings formed by objects for B^{n-1}, B^n et B^{n+1}.

Corollary. *LaxLink* is a left inverse of a sub-functor *Cyl* of *Cub,* so that an *n*-fold category **A** is isomorphic to *LaxLink(Cyl***A***)***.**

*Cyl***A** is the *n*-fold subcategory of *Cub***A** formed by the cubes whose source and target for the *n*-th law are also objects for the *n*-1-th law.



Theorem. Cat_n admits a monoidal closed structure whose internal Hom is

$$\text{LaxHom}_n(\mathbf{A}, \mathbf{B}) = \mathbf{MCat}(\mathbf{A}, (Cub^n\mathbf{B})^{1,3,\ldots,2n-1,\ 2,4,\ldots,2n})$$

The tensor of (**A, B)** is $LaxLink^n(\mathbf{A} \lozenge \mathbf{B})^{1,n+1,,\ldots,i,n+i,,\ldots,n,2n}$ and its unit is the *n*-fold category on **1**.

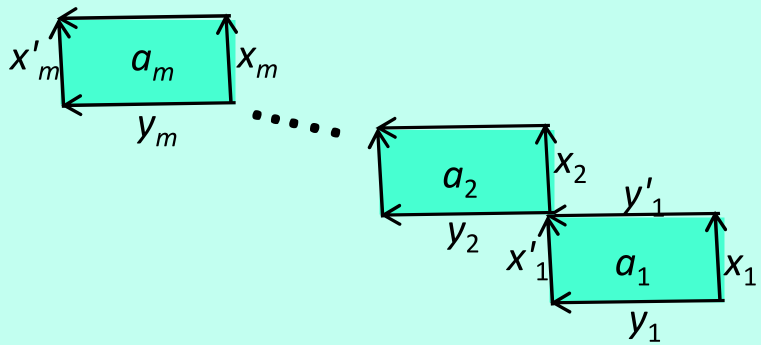
RELATIONS BETWEEN $\mathbf{Cat}_n$ AND 2-CATEGORIES

Given a 2-category **C**, we denote by Q(**C**) the double subcategory of *Cub***C** formed by the objects for the first law, i.e. by the double functors **Q** → **C** (often called squares of **C**, or, by Charles, *quintets* of **C**).

Theorem. The insertion of the category 2-Cat of 2-categories into Cat_2 has a left adjoint *String* such that each double category **A** is isomorphic to a double subcategory of the double category Q(*String***A**), where

$$\mathit{String}\mathbf{A} = \mathit{LaxLink}\mathrm{A}^{\mathrm{dis},1,2}.$$

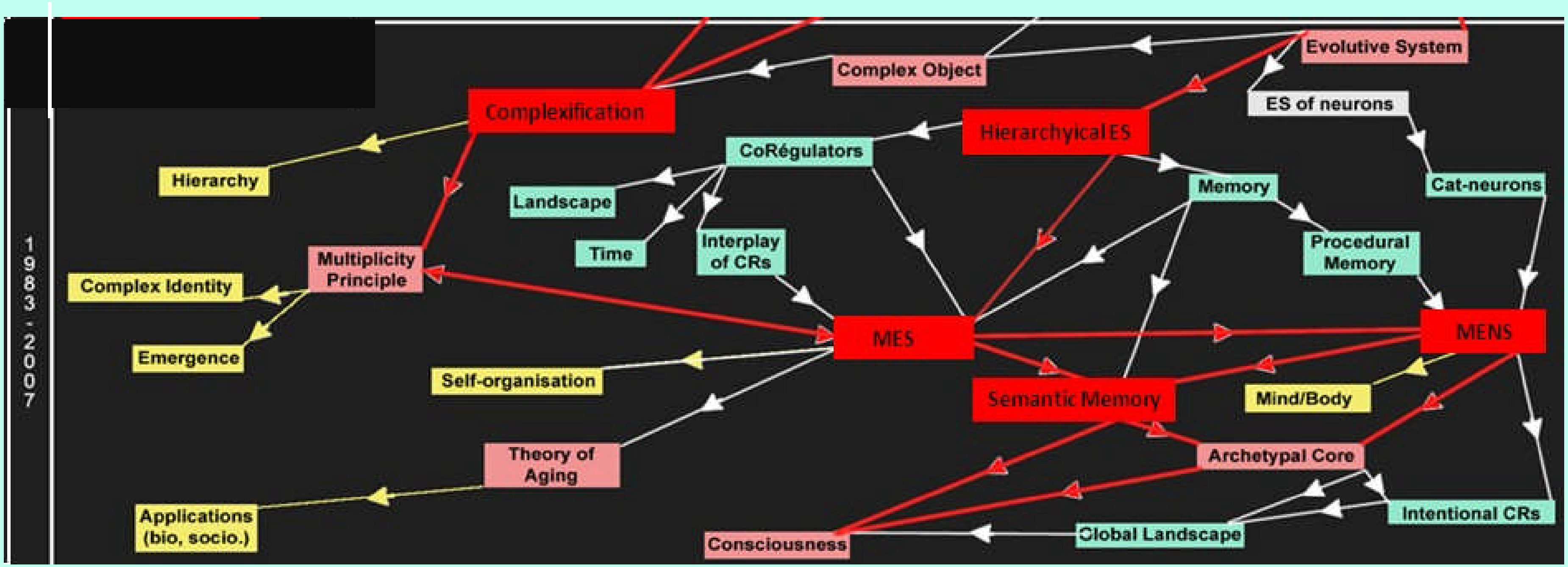
The 1-morphisms of *String***A** are (classes of) strings of objects for alternately the 1st and the 2nd law, and a 2-cell is a string of blocks $(a_n,\ldots,a_1)$.



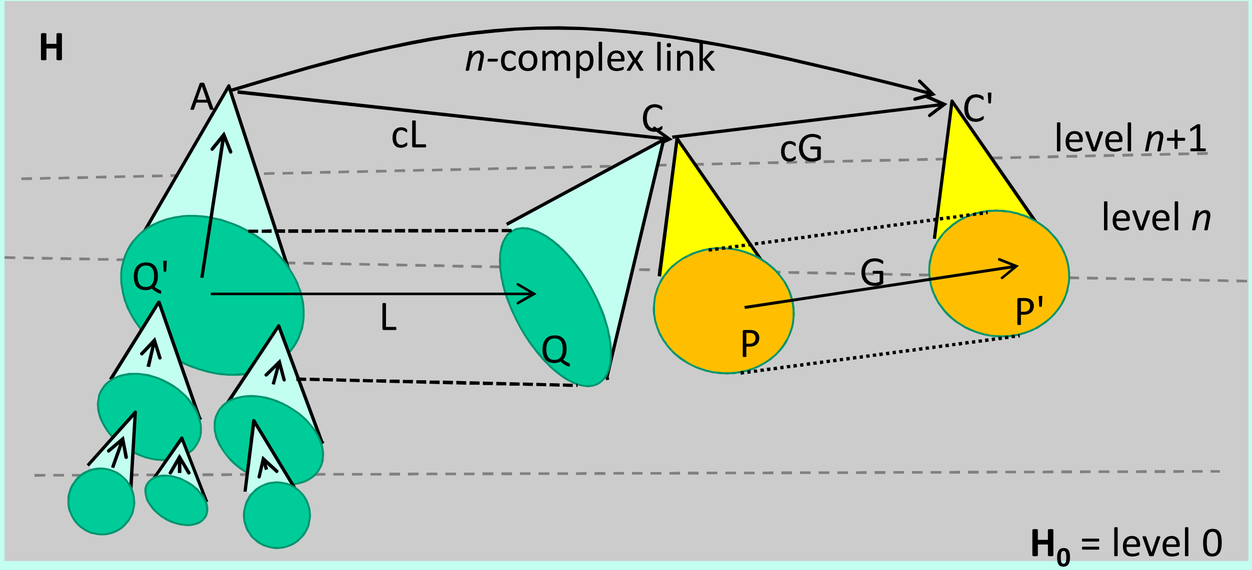
(Guitart proved that it follows that **A** is also isomorphic to a double subcategory of Q(**Cat**).)

Theorem. For $n > 2$, an n-fold category **A** admits an n-fold embedding into the n-fold category Cub^{n-2}Q(**C**), where **C** is the 2-category *String*($\mathit{LaxLink}^{n-2}$A).

A NEW ORIENTATION
MODELING LIVING SYSTEMS



HIERARCHICAL CATEGORIES

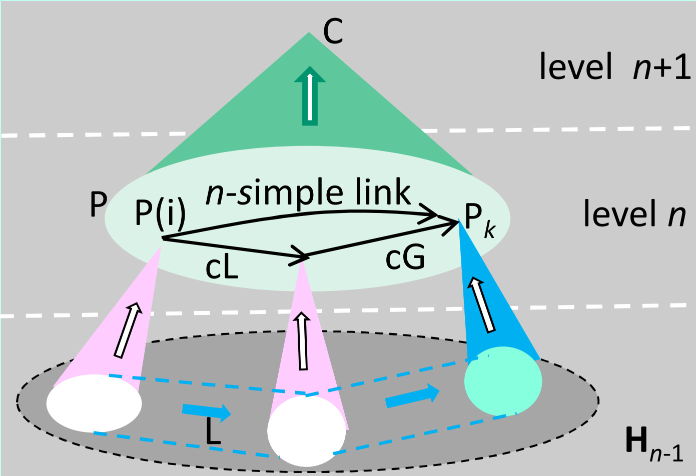


A category **H** is *hierarchical* if its objects are partitioned into 'levels', so that each object A of level *n+1* is the colimit of at least one pattern (= diagram) with values in the full sub-category $\mathbf{H}_n$ whose objects are of level ≤ *n.* Then A admits 'ramifications' down to level 0.

A pattern P in **H** defines an object P* of the category Ind**H**. A morphism G*: P* → P'* in Ind**H** corresponds to a *cluster* G of morphisms between components of P and P'. If P and P' are patterns in $\mathbf{H}_n$ admitting colimits C and C' in **H**, then G 'binds' into a morphism cG: C → C', called an *n-simple link.* The composite cG cL of *n*-simple links may not be *n*-simple if C is the colimit of two patterns P and Q in $\mathbf{H}_n$ such that P* and Q* are not isomorphic in Ind**H**. Such a C is called a *multiform object,* and cG cL is an *n-complex link.* These links model *n*+1-*emergent properties.*

H is a *based hierarchy* if all the morphisms are *n*-simple or *n*-complex for some *n.*

COMPLEXITY ORDER. COMPLEXIFICATION



An object C of **H** of level n+1 is *m-reducible* if it is the colimit of a pattern in $\mathbf{H}_m$. The *complexity order* of C is the smallest $m \leq n$ such that C is m-reducible.

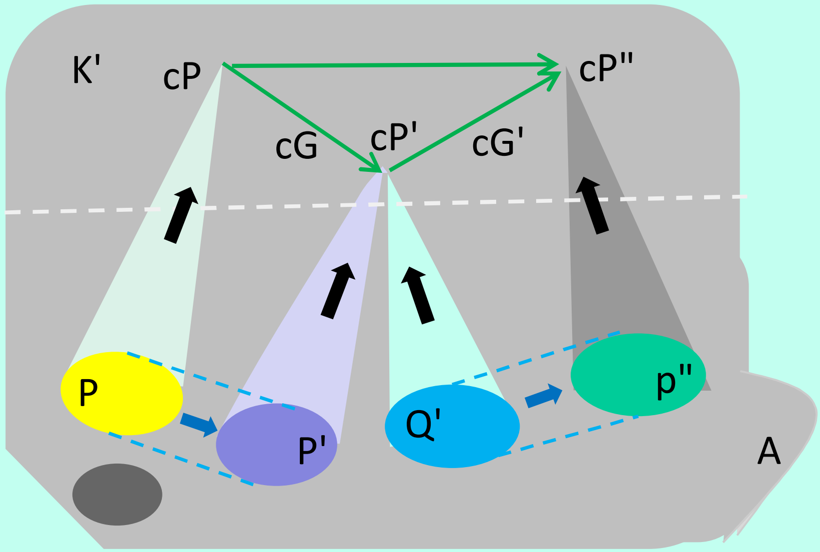
Theorem. Let C be an object of level n+1 which is the colimit of a pattern P in $\mathbf{H}_n$. If no P(i) is multiform, C is n-1-reducible, hence of complexity order $\leq n$-1.

Corollary. In a hierarchical category without multiform objects, all objects are of complexity order 0.

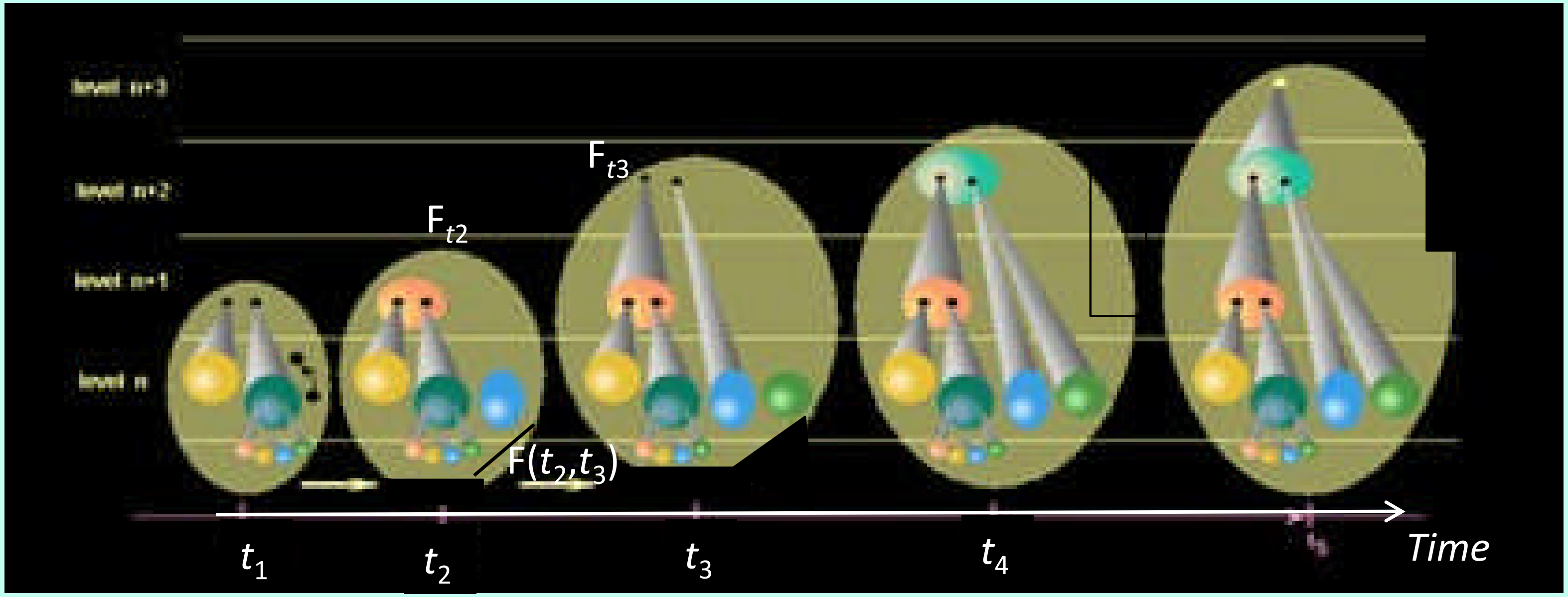
A based hierarchy **H** is consructed from $\mathbf{H}_0$ through iterated complexifications just adding colimits.

A *complexification* K' of a category K has for objectives to 'suppress' a given set E of objects, to 'absorb' a category A, to 'add' (co)limits to some patterns and to preserve some given (co)limits. It is constructed as the prototype associated to a sketch.

Theorem. If K has multiform objects, so does a complexification K' of K, and iterated complexifications lead to the emergence of objects of strictly increasing complexity order.



MEMORY EVOLUTIVE SYSTEMS

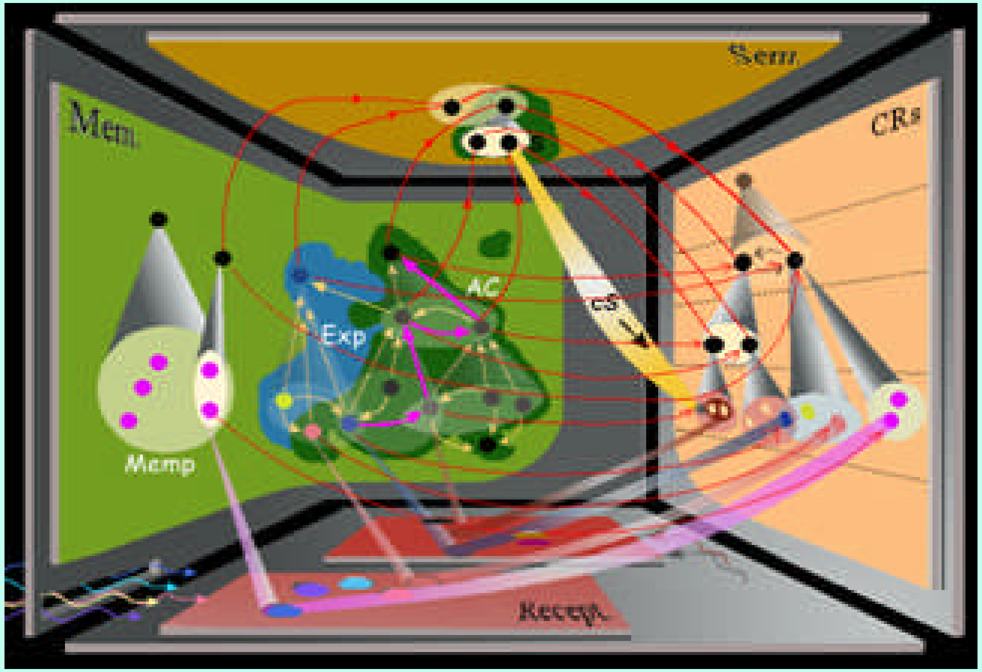


A *Hierarchical Evolutive System* is a partial action F of a category *Time* enriched in the category of hierarchical categories; it models the successive configurations of a complex natural system with a hierarchy of components of increasing complexity order. The 'transition' $F(t_i, t_j)$ models the change of configuration and is obtained by a complexification process, so that:

Theorem. The emergence of objects of strictly increasing complexity order is characterized by the existence of multiform objects (*Multiplicity Principle*).

To model self-organization, we define a *Memory Evolutive System* as a HES with a net of subsystems, the *Coregulators*, modeling specialized internal regulation organs, and a sub-system, the *Memory*, developed by learning. The dynamics is modulated by the competitive interactions between CRs, each CR selecting objectives on its 'landscape' with its own timescale.

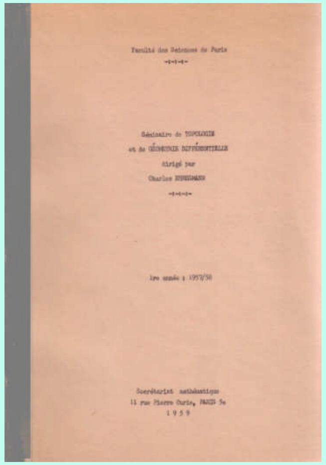
THE MENS MODEL FOR COGNITIVE SYSTEMS

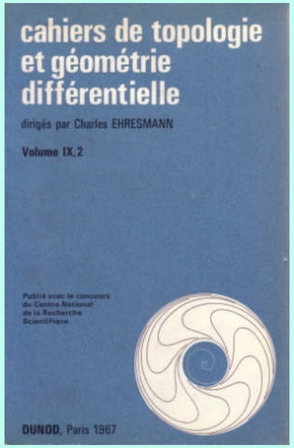


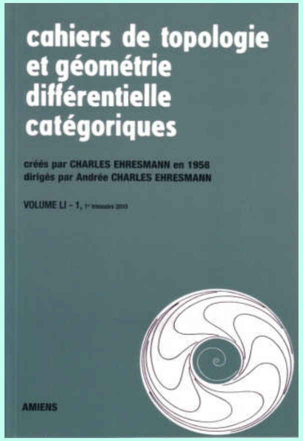
The model MENS (Memory Evolutive Neural System) of a neural and cognitive system is a MES obtained by successive complexifications of the ES of neurons modelling the neural system of an animal. We describe how a procedural and a semantic memory develop and lead to the formation of an interconnected personal memory, the *archetypal core* essential for the emergence of higher cognitive processes up to conscious processes.

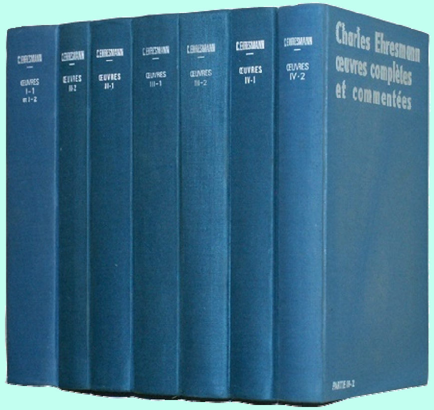
In MENS, the categories are equipped with a functor *d* to $\mathbf{R}_+$ which models the propagation delays of the morphisms. We characterize the 'polychronous' neural groups (Izhikevitch & al.) activated by mental events by a categorical property (*d-connexity)*, which allows the extension of *d* to a complexification in which the distinguished patterns are of this form. It allows extending to MENS the equations known for neural systems.

"CAHIERS" and CHARLES' OEUVRES









The "Cahiers" began under the title "Séminaire de Topologie et Géométrie Différentielle" in 1957/58, published by Belgodère. To be more free, Charles decided to publish them directly and I helped with the material work. They became the quarterly journal "Cahiers de Topologie et Géométrie Différentielle" in 1967 and were briefly edited by Dunod up to 1972, when we took the entire editing work. In 1984, I added the word "Catégoriques" and created an Editorial board. Now the "Cahiers" have an Internet site, and backsets are made available by NUMDAM.

From 1980 to 1983, the 7 volumes of "Charles Ehresmann : Œuvres complètes et commentées" appeared as Supplements. They contain all Charles' papers and also about 450 pages of comments (in english) to translate' them is a more modern language, to replace them in the historical context, to extend some results. enrichment