From Schwartz Distributions to Control and Evolutive Systems

by

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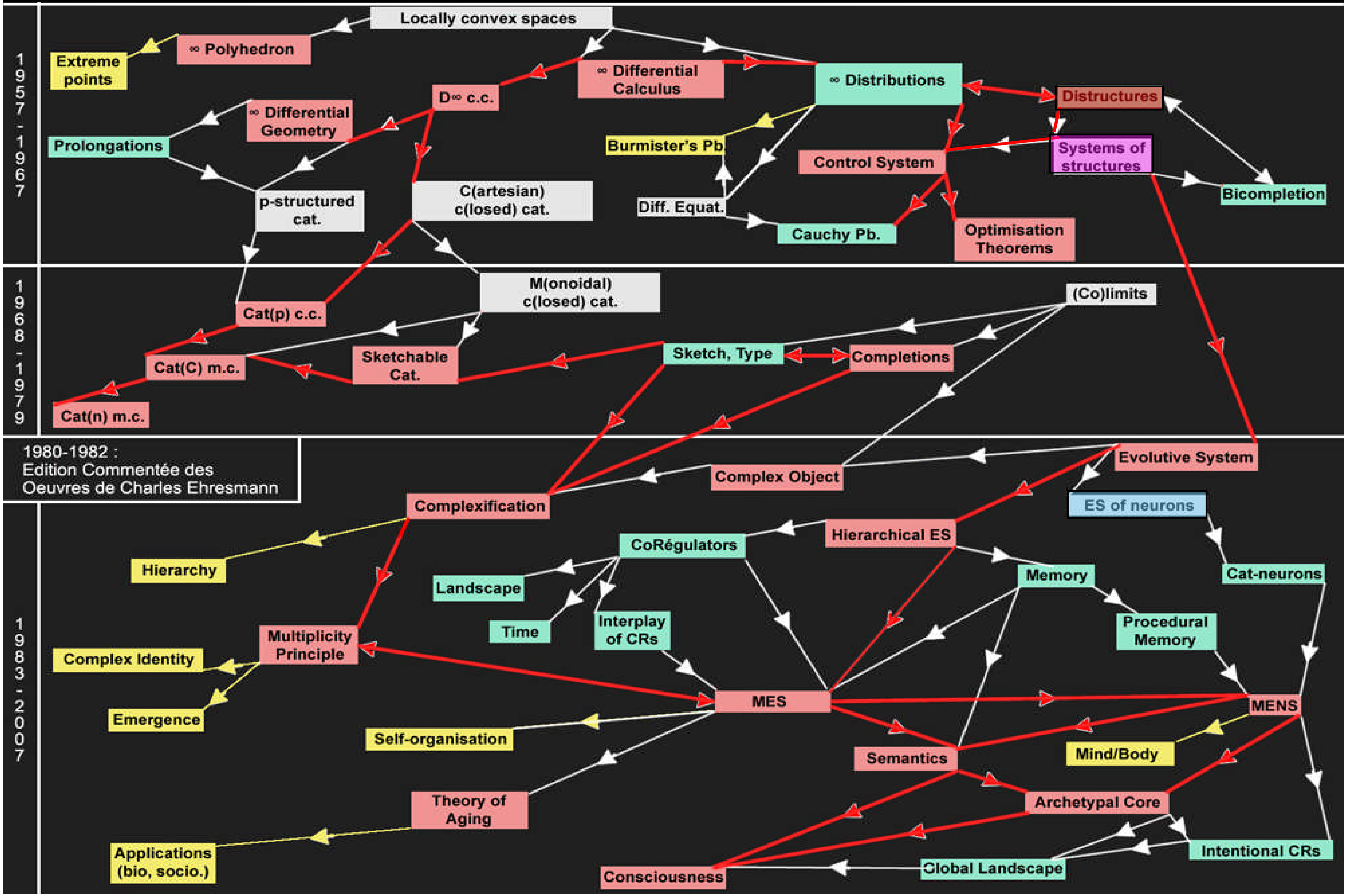
http://pagesperso-orange.fr/vbm-ehr/ACE/ehr.htm

DIAGRAM OF MY RESEARCH WORK



PART 1: SEMI-SHEAVES

1.1. SEMI-SHEAVES OF SETS

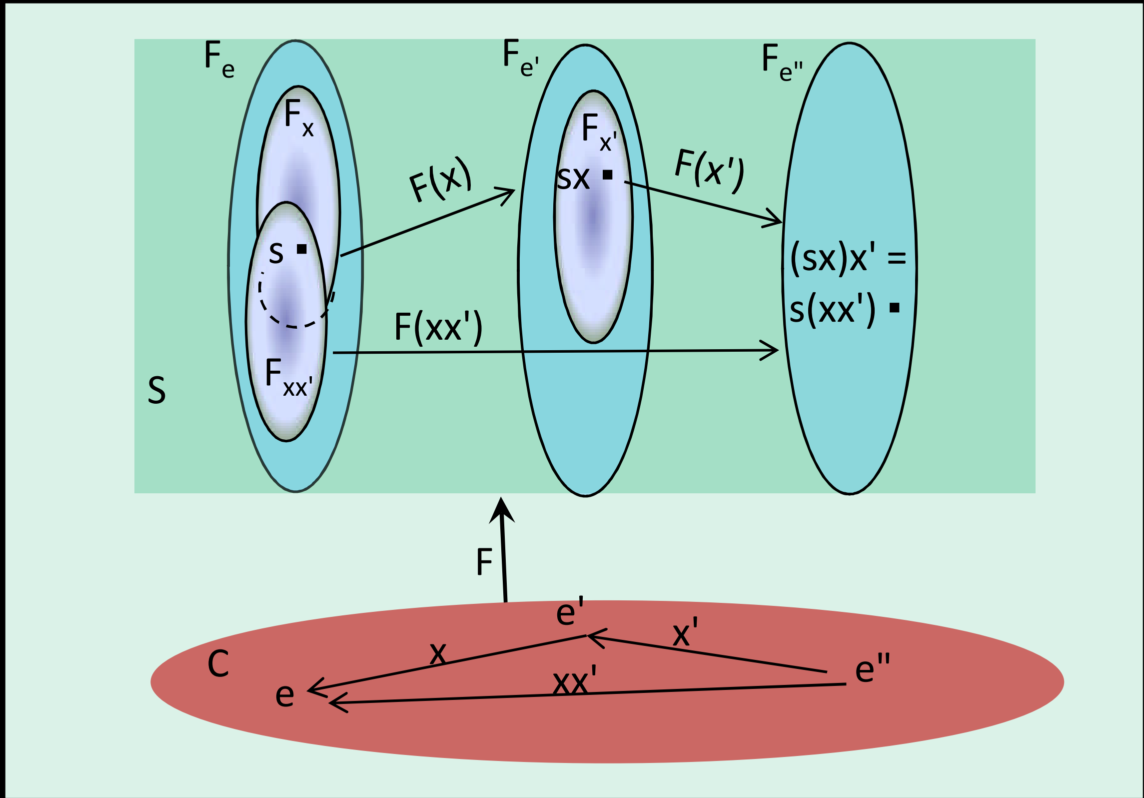
1.2. (K, M)-SEMI-SHEAVES

1.3. THE CATEGORY SS(K, M)

1.4. K-PRESHEAFIFICATION OF A (K, M)-SEMI-SHEAF

1.5. LOCALLY CONVEX SEMI-SHEAVES

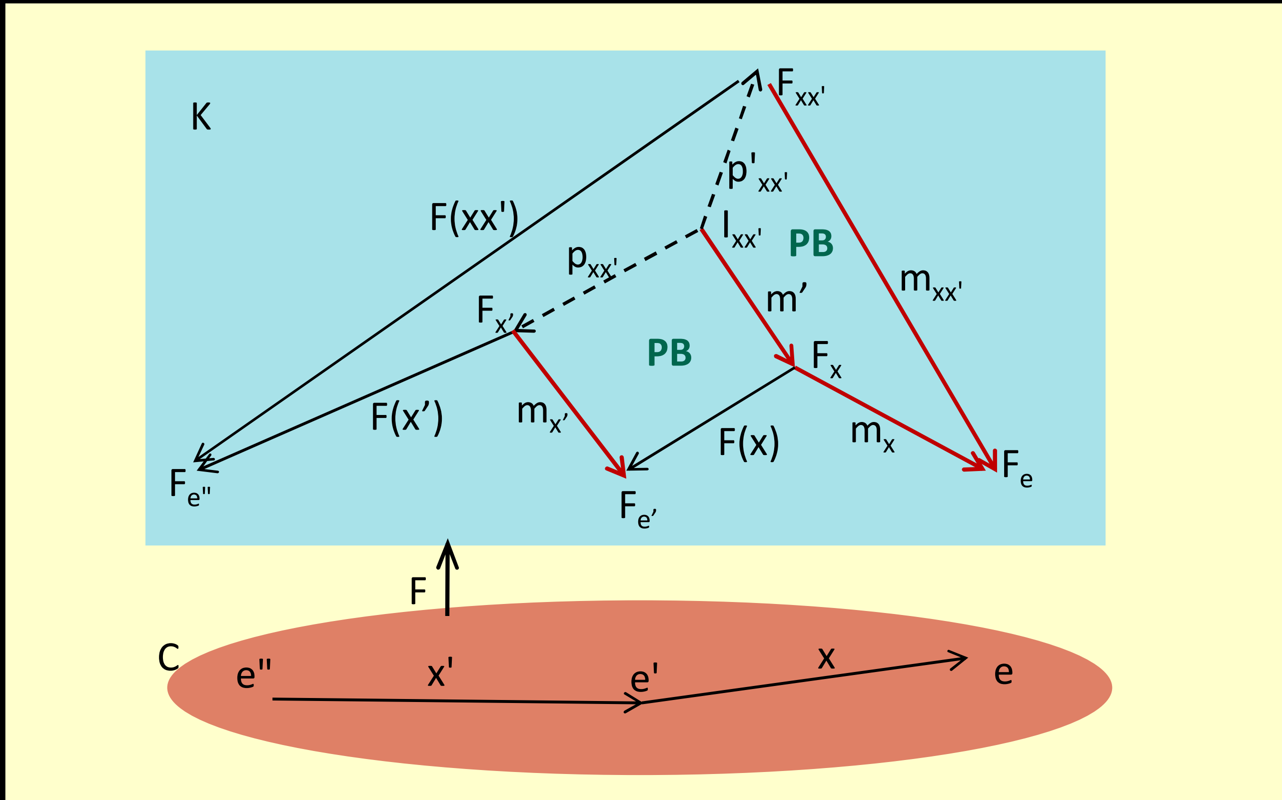
1.1. SEMI-SHEAVES OF SETS



A *semi-sheaf (of sets)* F on C is a map from C to Set which associates to an object e of C the fibre F_e on e, to an arrow x from e' to e a map F(x) from a sub-set F_x of F_e to the fibre on e', and satisfies the 'transitivity condition': if s is in F_x then sx = F(x)(s) is in $F_{x'}$ if and only if s is also in $F_{xx'}$, and then s(xx') = (sx)x'.

We also call the map (x, s) |→ sx if s is in F_x a *semi-action* of C^{op} on the disjoint union S of the fibers. (The *system of structures* of C. Ehresmann are a slightly more general notion.)

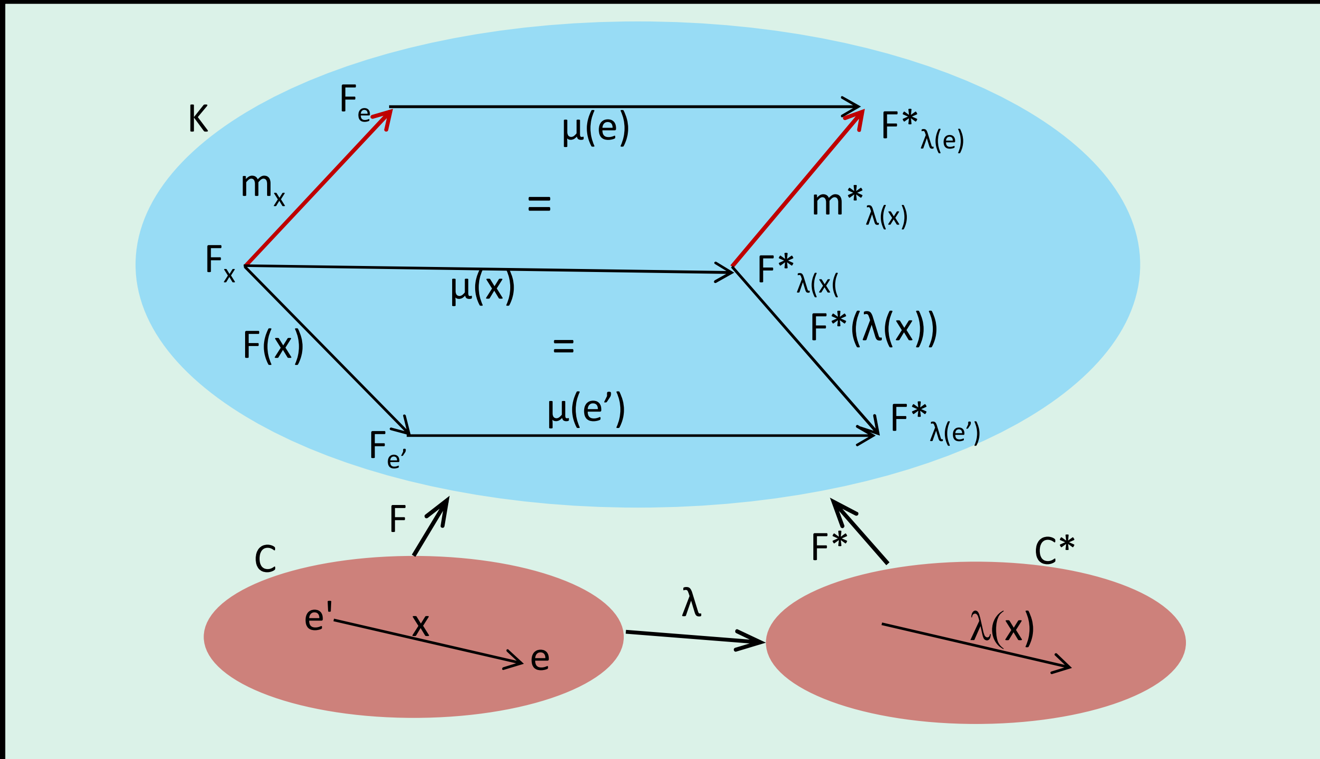
1.2. (K, M)-SEMI-SHEAVES



K is a category and M a class of monomorphisms (in red on the figure) of K containing the identities, stable by pullbacks and with at most one m between 2 objects. A (K, M)-*semi-sheaf* F on C is a map from C to K such that:

1. F(e) is an object F_e for each object e of C. 2. For x: e' → e, F(x): $F_x \to F_{e'}$ where F_x is an object such that there exists an m_x: $F_x \to F_e$ in M. 3. '*Transitivity condition*': for x': e'' → e' in C, the pullback $I_{xx'}$ of ($m_{x'}$, F(x)) is also the pullback of (m_x, $m_{xx'}$) and $F(xx')p_{xx'} = F(x')p'_{xx'}$ where $p_{xx'}$ and $p'_{xx'}$ are projections of the PB's.

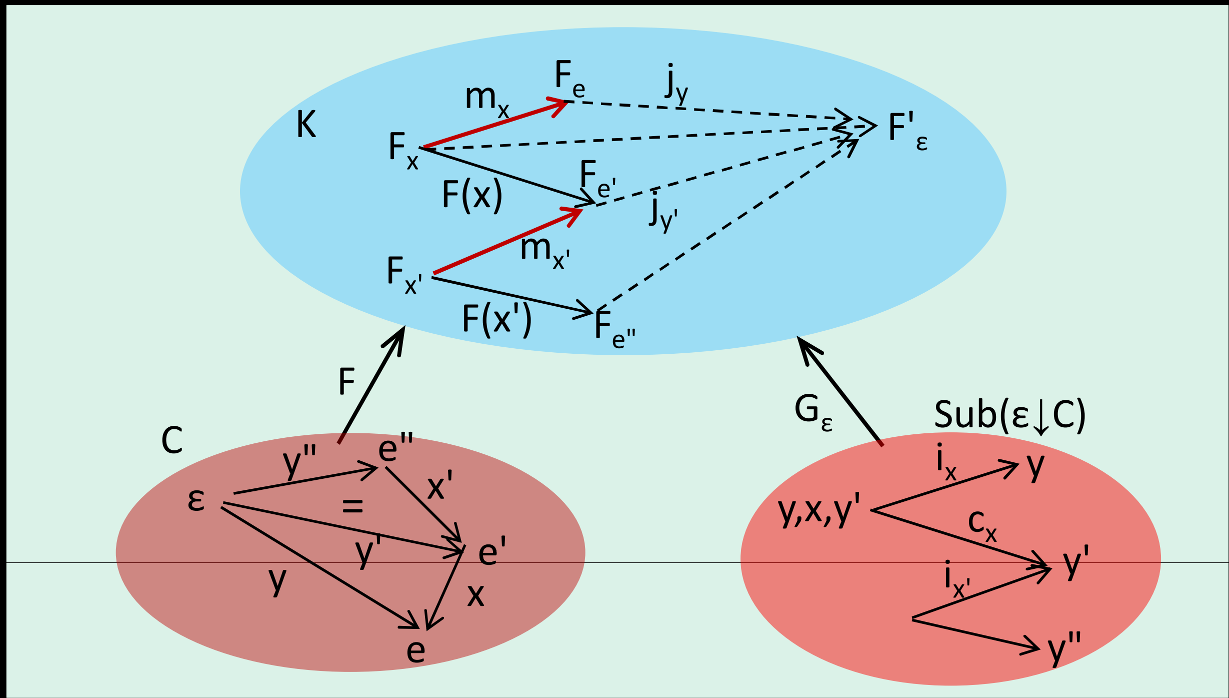
1.3. THE CATEGORY SS(K, M)



If F is a (K, M)-semi-sheaf on C and F* a (K, M)-semi-sheaf on C*, we define a morphism (λ, μ): F → F* by a functor λ from C to C* and a map μ from C to K making commutative the above diagrams. The category SS(K, M) admits as a full subcategory the category PS(K) of K-presheaves.

Examples. I have used particular (Top, open ins.)-semi-sheaves, called *nuclei of actions*, to study control problems for differential equations. The Evolutive Systems (defined with J.P. Vanbremeersch for modeling living systems) are (Cat, subcat)-semi-sheaves.

1.4. K-PRESHEAFIFICATION OF A (K, M)-SEMI-SHEAF



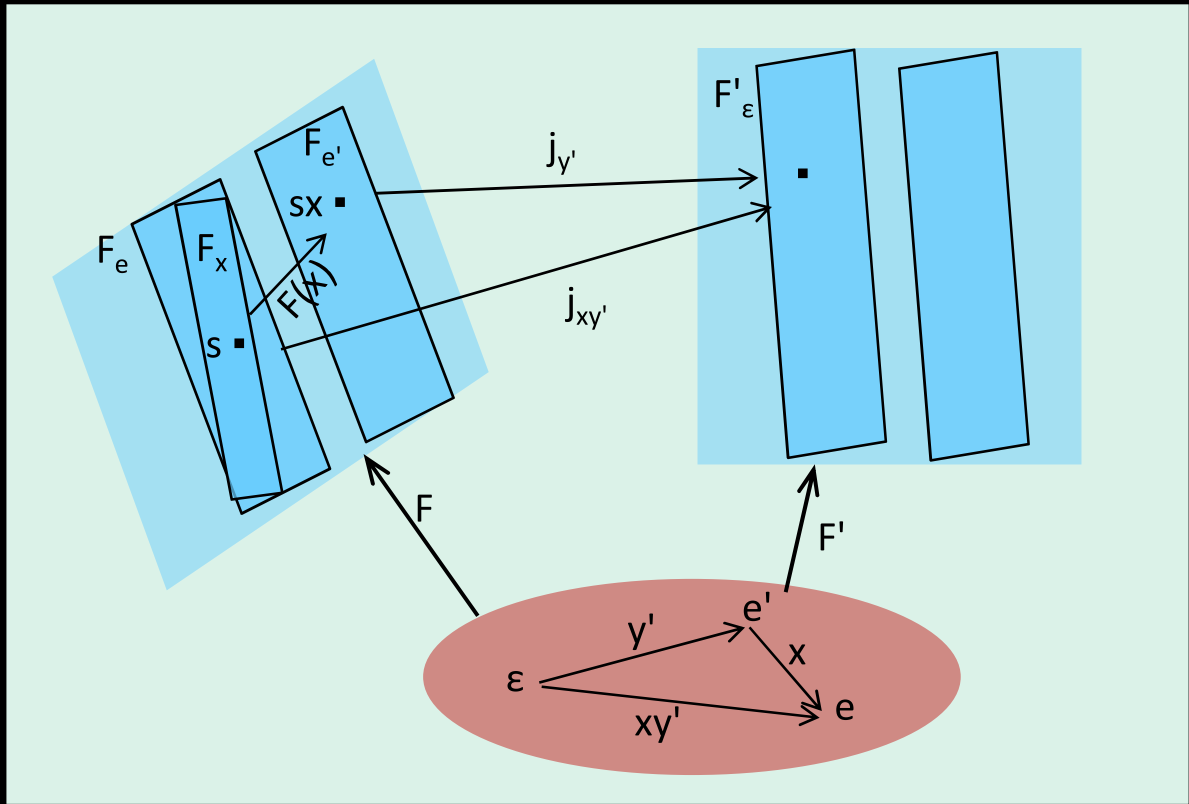
Associated Presheaf Theorem. *If* K *admits colimits, the category* PS(K) *of* K*-presheaves is a reflective subcategory of the category* SS(K, M).

The K-presheaf F' associated to the (K, M)-semi-sheaf F is explicitly constructed: its fiber F'_ε is the colimit of the functor G_ε: Sub(ε↓C) → K where Sub(ε↓C) is the subdivision category of the category ε↓C of objects under ε,

$$G_\varepsilon(i_x) = m_x \quad \text{and} \quad G_\varepsilon(c_x) = F(x).$$

If K = Set, this F'_ε becomes the quotient of the sum of $(F_e)_{y:\varepsilon\to e}$ by the equivalence generated by the relations: (y, sx) ~ (xy, s) for each s in F_x.

1.5. LOCALLY CONVEX SEMI-SHEAVES



If K is the category Lcs of locally convex spaces and M_V the class of insertions of a vector subspace with a finer topology, an (Lcs, M_V)-semi-sheaf F is called a *locally convex semi-sheaf,* and its associated Lcs-presheaf F' a l*ocally convex presheaf*. The fiber F'_ε of F' on ε (colimit of G_ε) is the lcs space quotient of the locally convex direct sum of the family of lcs spaces $(F_e)_{y:\varepsilon \to e}$ by the equivalence relation generated by the relations: (y', sx) ~ (xy', s) where s is in F_x and x: e' → e in C and sx = F(x)(s).

PART 2 : DISTRUCTURES

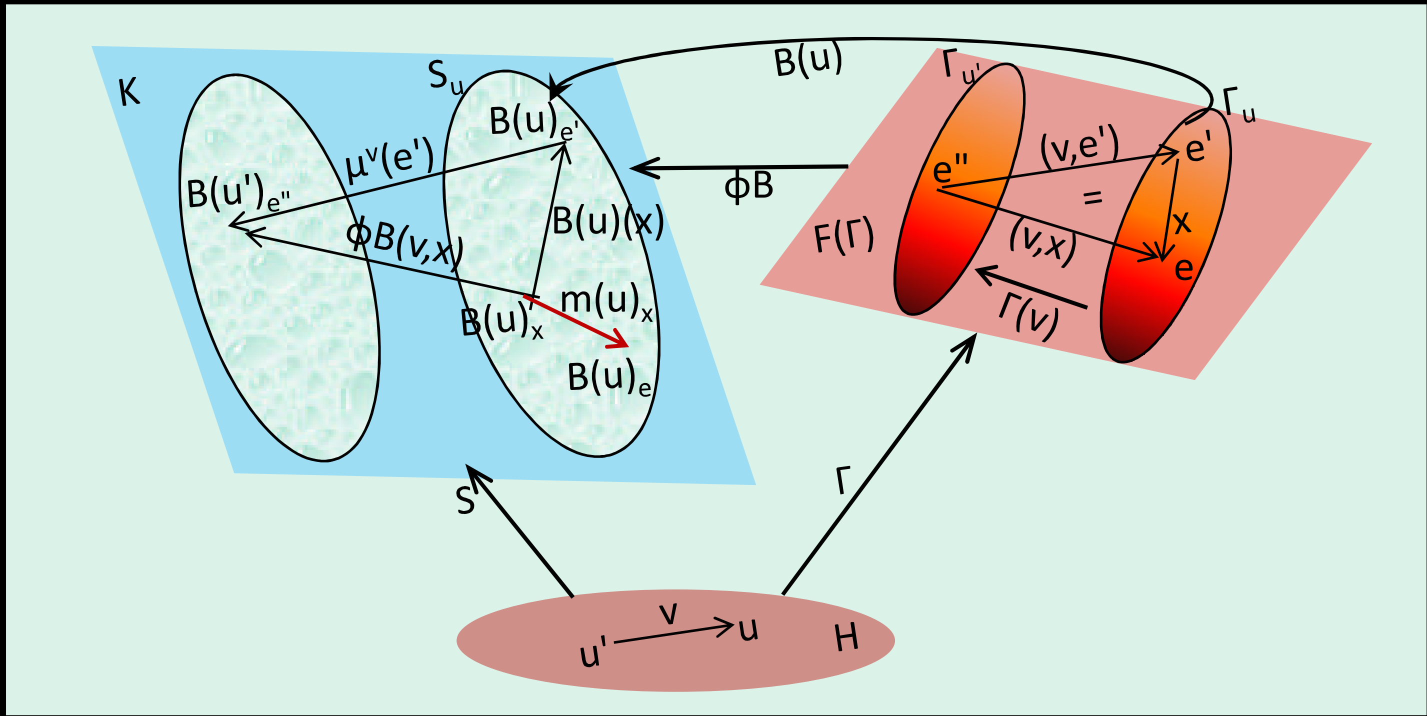
2.1. GENERATOR OF (K, M)--DISTRUCTURES

2.2. ANOTHER CHARACTERIZATION OF DISTRUCTURES

2.3. LOCALLY CONVEX DISTRUCTURES

2.4. A (PRE)SHEAF OF DISTRUCTURES OVER MONOIDS

2.1. GENERATOR OF (K, M)-DISTRUCTURES

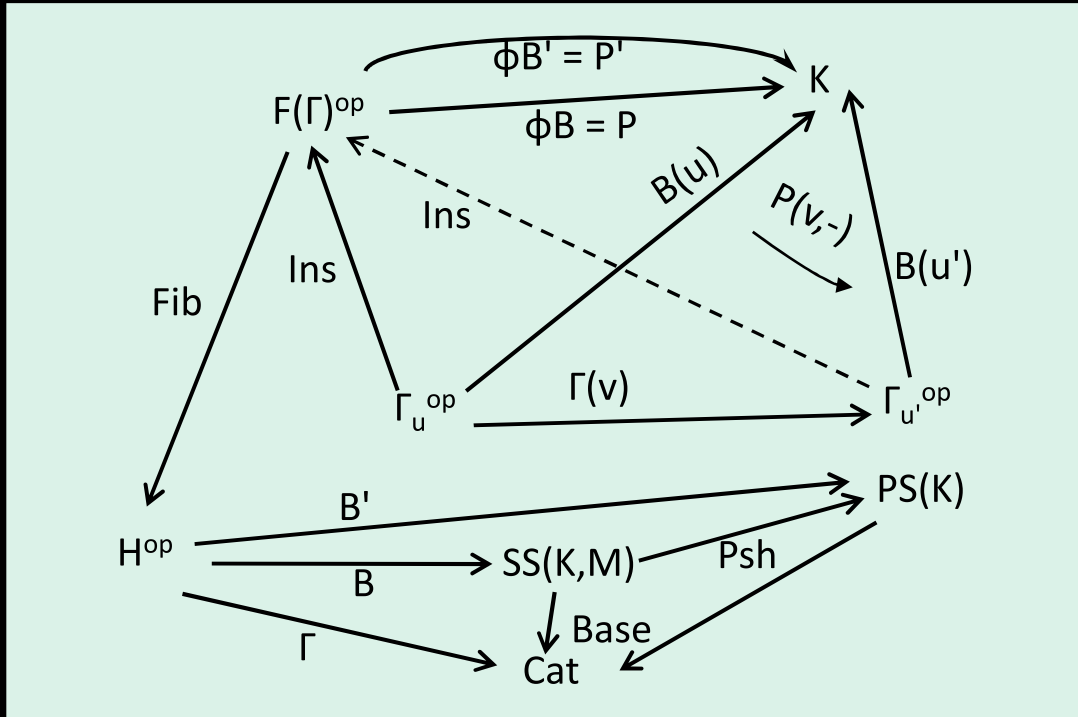


A *generator of* (K,M)-*distructures* on a category H is a functor B from H^{op} to the category SS(K M) of (K, M)-semi-sheaves. Thus B(u), for an object u of H, is a (K, M)-semi-sheaf on a category Γ_{u}, and B(v) is a morphism (Γ(v), μ^v): B(u) → B(u'). Its composite B' with the associated presheaf functor Psh from SS(K, M) to PS(K) is called the *presheaf of* K-*distructures generated by* B.

The Γ_u are the fibers of a presheaf of categories Γ on H; let F(Γ) be its associated fibration. B determines a (K, M)-semi-sheaf φB on F(Γ) with $B(u)_e$ as its fiber on (u, e),

$$\varphi B_{(v,,x)} = B(u)_x \quad \text{and} \quad \varphi B(v, x) = \mu^v(e')B(u)(x).$$

2.2. ANOTHER CHARACTERISATION OF DISTRUCTURES

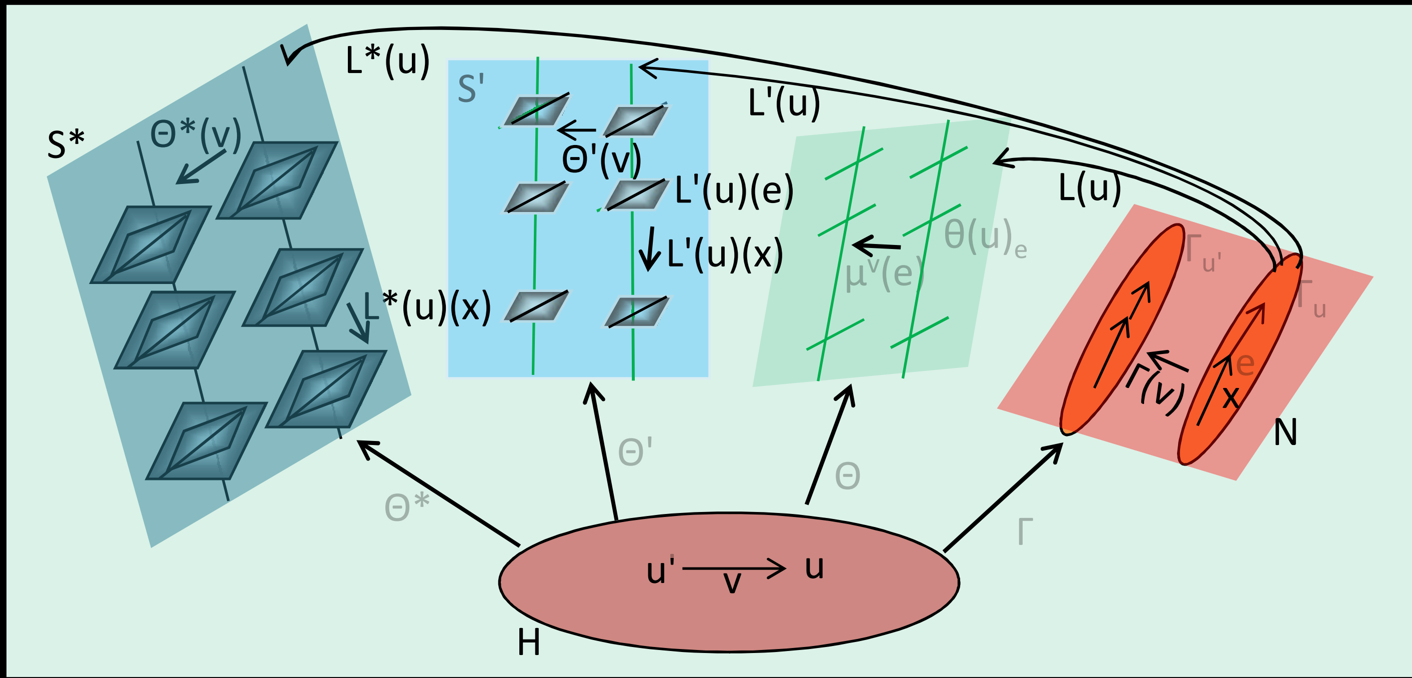


Theorem. *There exists a 1-1 correspondance* φ *between the generators* B *of* (K, M)-*distructures with* Γ *as their base* Cat-*presheaf and the (*K, M)-*semi-sheaves on the fibration* F(Γ) *associated to* Γ. *It sends the* K-*presheaf* B' *generated by* B *to the* K-*presheaf associated to* φB.

A (K,M)-semi-sheaf P on F(Γ) is of the form φB for the generator of (K, M)-distructures B such that B(u) is the composite of P with the insertion of Γ_u in F(Γ), and

$$B(v) = (\Gamma(v), P(v,-)) \quad \text{for} \quad v: u' \to u.$$

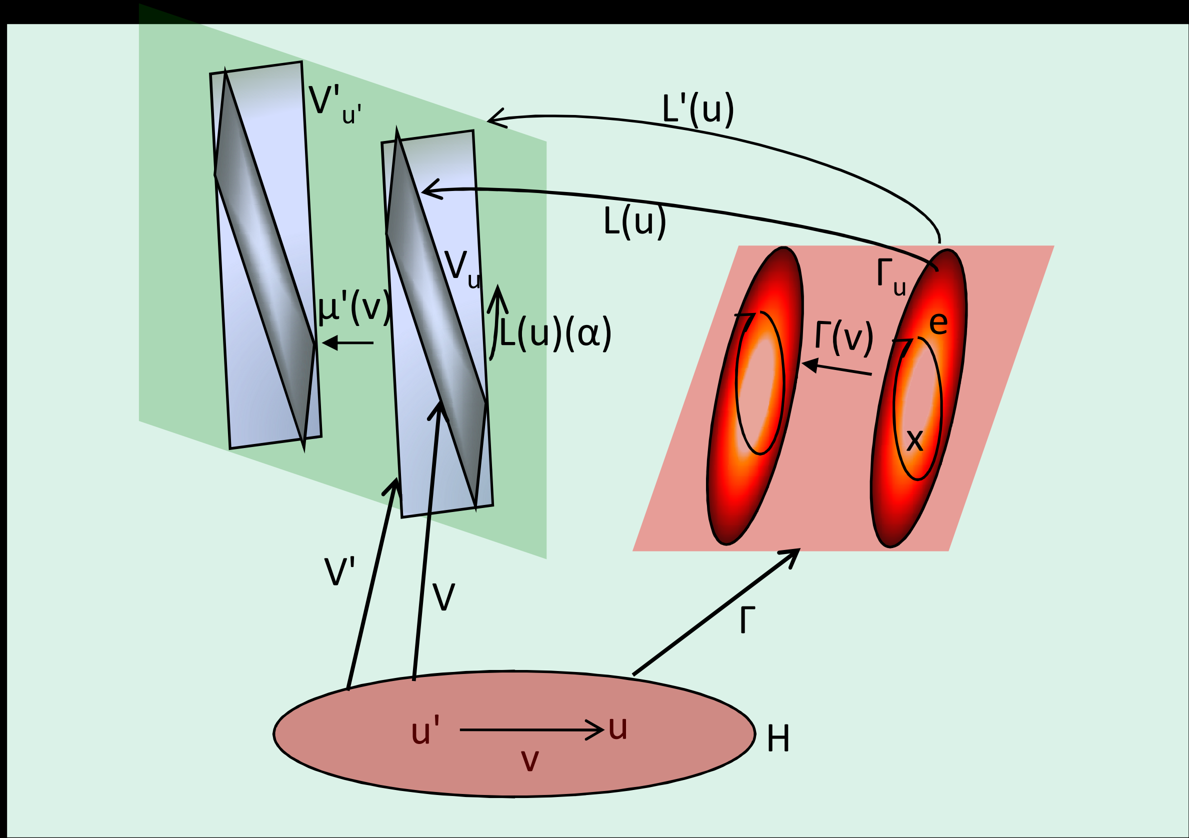
2.3. LOCALLY CONVEX DISTRUCTURES



A *generator of locally convex distructures* L, and its generated presheaf L', are obtained in the case (K, M) = (Lcs, M_V). Then L and L' have underlying Top-presheaves Θ and Θ', with Θ_u = coproduct of the topologies on the fibers of L(u). The figure emphasizes the 'di-structure' on the topological sum S' of the fibers of Θ': the category H^{op} acts 'horizontally' on it while N^{op} acts 'vertically', where N = coproduct of the Γ_u.

If H is equipped with a topology of Grothendieck T, and if Γ a sheaf for T, the *sheaf of distructures generated by* L (or by L') is the composite L* of L' with the associated Lcs-sheaf functor Sh. The Top-sheaf Θ* underlying L* is the Top-sheaf associated to Θ'. The Lcs-presheaf φL* corresponding to L* is a sheaf for the topology on F(Γ) 'lifted' from T.

2.4. A (PRE)SHEAF OF DISTRUCTURES OVER MONOIDS



If L is a generator of locally convex distructures such that the base Cat-presheaf Γ is a sheaf of monoids, then L(u) has only one fibre $L(U)_e = V_{u.}$.

This V_u is also the fiber of an Lcs-presheaf V on H having Θ as its underlying Top-presheaf. Thus L is entirely determined by: a presheaf Γ of monoids, a presheaf V of locally convex spaces and, for each u, a semi-action of Γ_u^{op} on V_u, these semi-actions being preserved by the change of fibers. Idem for the presheaf L' and the sheaf L* generated by L.

PART 3 : DISTRIBUTIONS

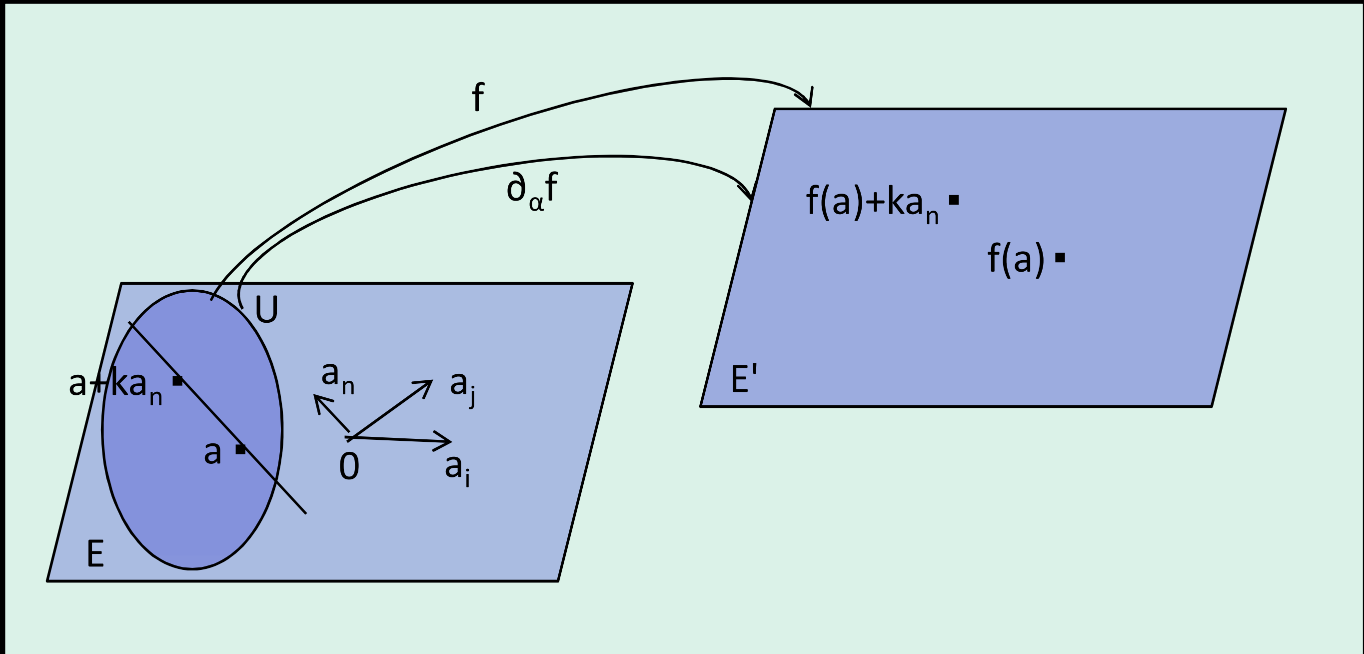
3.1. PARTIAL DERIVATIVES OF A FUNCTION

3.2. GENERATOR OF DISTRIBUTIONS

3.3. FINITE ORDER DISTRIBUTIONS

3.4. THE SHEAF OF DISTRIBUTIONS

3.1. PARTIAL DERIVATIVES OF A FUNCTION

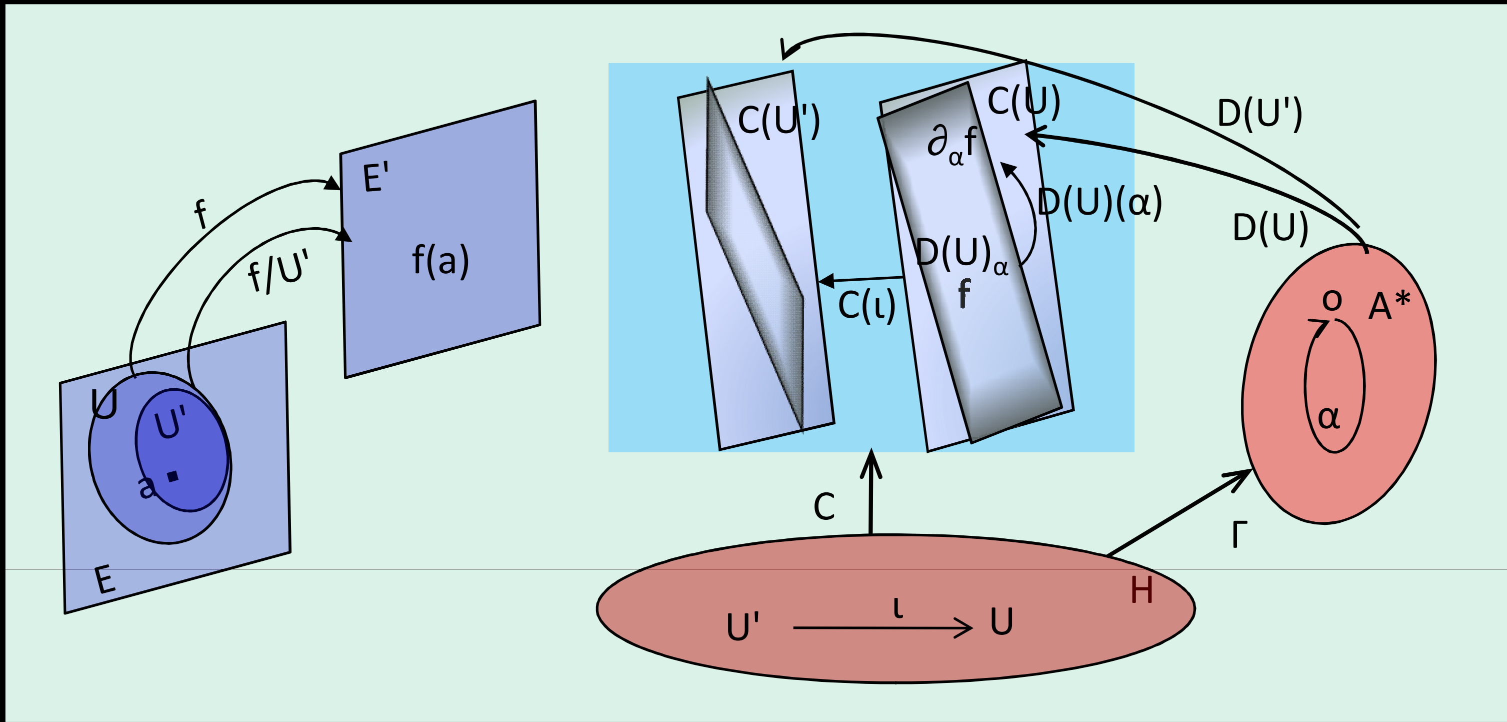


E is a locally convex space and A = $(a_i)_i$ an algebraic base of E. We denote by A* the free commutative monoid on A and by o its unit. An element α of A is a finite multiset α = {α_1, α_2,…α_n} of n (possibly repetitive) elements of A.

Let f be a function from an open subset U of E to a locally convex space E'. We say that f has an α-*derivative* $\partial_\alpha f(a)$ at a if the restriction of f to the affine sub-space $a+\Sigma_i \mathbf{R}\alpha_i$ is n-differentiable at a and admits $\partial_\alpha f(a)$ for its partial derivative with respect to (α_1, α_2, …α_n). (It is independent on the order.)

NB. The following theory of distributions could use another kind of differentiability.

3.2. GENERATOR OF DISTRIBUTIONS

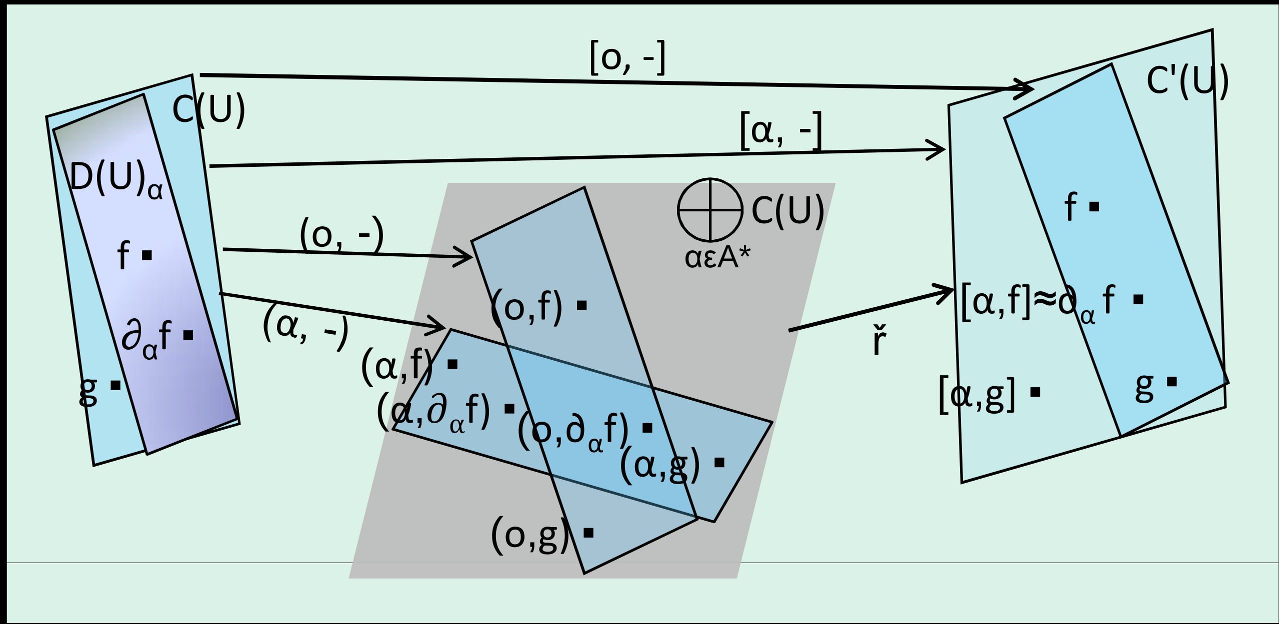


E and E' are complete metrizable locally convex spaces. A *generator of* E'-*valued distributions on* E is a generator D of locally convex distructures on the category H of open subsets U of E of the following form:

Its base Cat-sheaf Γ is the sheaf constant on the monoid A*; the unique fiber of D(U) is the lcs C(U) of continuous maps from E to E', with the compact-open topology; $D(U)_\alpha$ is the subspace of C(U) consisting of the functions f admitting a continuous α-derivative on U, with the compact-open topology for f and all its derivatives up to α, and D(U)(α)(f) = ∂_αf.

The underlying Lcs-presheaf C on H maps the insertion ι: U' → U on the restriction map g |→ g/U' from C(U) to C(U').

3.3. FINITE ORDER DISTRIBUTIONS



The presheaf D' generated by the generator of E'-distributions D is the *presheaf of* E'-*distributions of finite order*. Its fiber C'(U) at U is the lcs: C'(U) = $\bigoplus_{\alpha \varepsilon A^*}$C(U)/r, with r the equivalence generated by: (αα', f) ~ (α', ∂_αf) if f is in D(U)$_\alpha$.

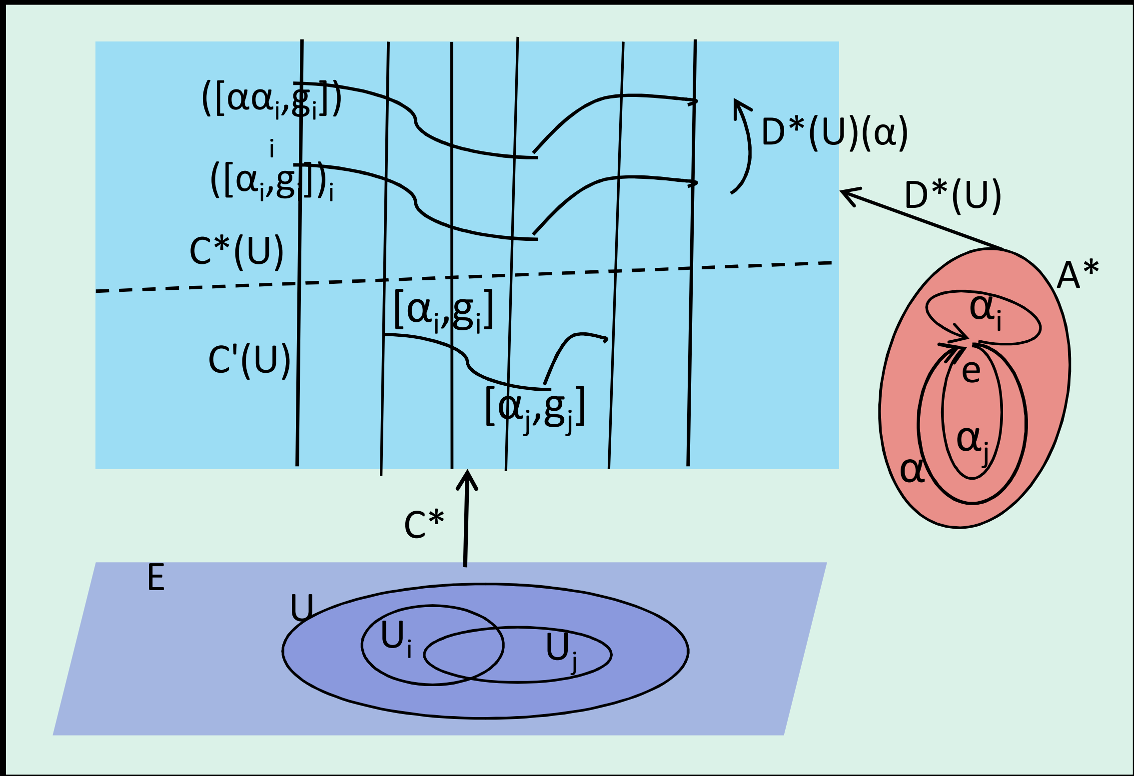
Theorem. *An element of* C'(U) *(or finite order distribution) is an equivalence class* [α, g] *for the equivalence on* A*×C(U): (α, g) ~ (α', g') *iff there exist*

β, β' *in* A* *and* g, g' *such that* **α**β = α'β', g = ∂_βf, g' = $\partial_{\beta'}$f' and $\partial_{\alpha\beta}$(g – g') = 0.

The topology on C'(U) *is the final locally convex topology for the maps* [α, -]: g |→ [α, g] *from* C(U) *to* C'(U). *The map* [o, -] *identifies* C(U) *to a vector subspace of* C'(U).

[The proof uses that each continuous function is of the form ∂_α f for some f in D(U) $_\alpha$.]

3.4. THE SHEAF OF DISTRIBUTIONS



The sheaf D* associated to D' is the *sheaf of* E'*-distributions* on E. If U is the union of $(U_i)_i$ an element of its fibre C*(U) is a family $([\alpha_i, g_i])_i$ drawn from C'(U) such that : $[\alpha_i, g_i/U_{ij}] = [\alpha_j, g_j/U_{ij}]$ on $U_{ij} = U_i \cap U_j$. And D*(U)(α) maps $([\alpha_i, g_i]_i)$ on $([\alpha\alpha_i, g_i])_i$.

If E is a finite dimensional space, the E'-distributions coincide with the Schwartz E'-valued distributions (via the weak integral $[\alpha, g]\varphi = (-1)^n \int g \partial_\alpha \varphi$ if α has n terms.and φ is a C^∞ function from E to **R** with compact support).

PART 4 : COMPLEMENTS

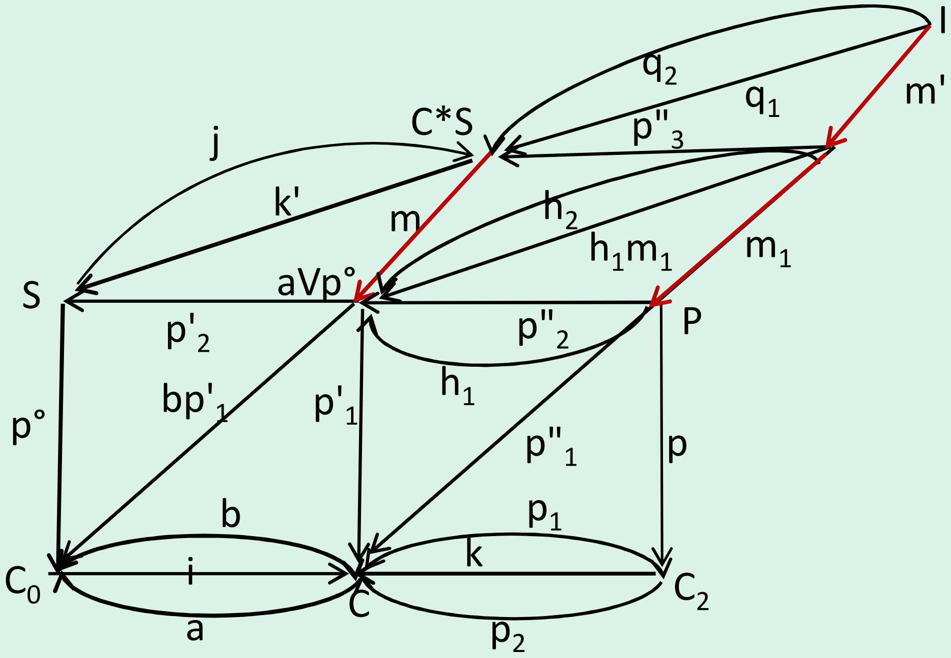
4.1. INTERNAL SEMI-SHEAVES

4.2. CONTROL SYSTEMS

4.3. HIERARCHICAL EVOLUTIVE SYSTEMS

4.4. THE MODEL MENS FOR COGNITION

4.1. INTERNAL SEMI-SHEAVES



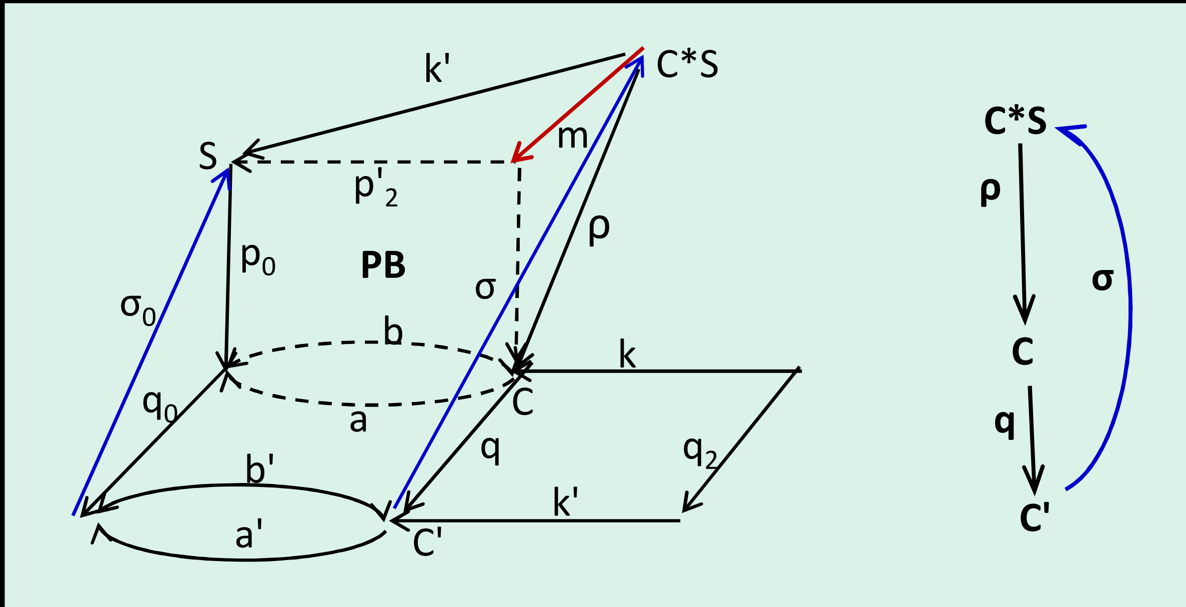
6 PB: $ap_1 = bp_2$, $ap'_1 = p°p'_2$, $ap''_1 = bp'_1p''_2$, $p''_2m_1 = mp$
$mq_1 = h_1m_1 m'$, $mq_2 = h_2m'$,
Axioms: $ak = ap_2$ and $bk = bp_1$
$p°k' = bp'_1m$ (action)
$p'_1mj = ip°$ and $p'_2mj = id_S = k'j$
$p_1p = p''_1$ and $p_2p = p'_1p''_2$
$p'_1h_1 = kp$ and $p'_2h_1 = p'_2p''_2$
$p'_1h_2 = p''_1m_1$ and $p'_2h_2 = k'p''_3$
$k'q_1 = k'q_2$ (transitivity)

To define internal semi-sheaves, we construct the above sketch of the (equivalent notion of) semi-action k': C*S → S of a category C on S; the distinguished cones are the 6 PB. A *semi-sheaf internal to* (K, M) on the category **C**^{op} internal to K is a model R of this sketch in K mapping m in M; it defines an *internal semi-action* R(k') of **C** on **S** = R(S), and a *semi-fibration* R(p'_1m): **C*S** → **C**.

Theorem. *If* K *admits pullbacks and cokernels stable by pullbacks, the Associated Presheaf Theorem extends to this internal case*.

The internal to K action associated to R is an action **k"** of **C** on the cokernel **S'** of an adequate pair (R(m), n), with **k"** deduced from R(h_1) via cokernels.

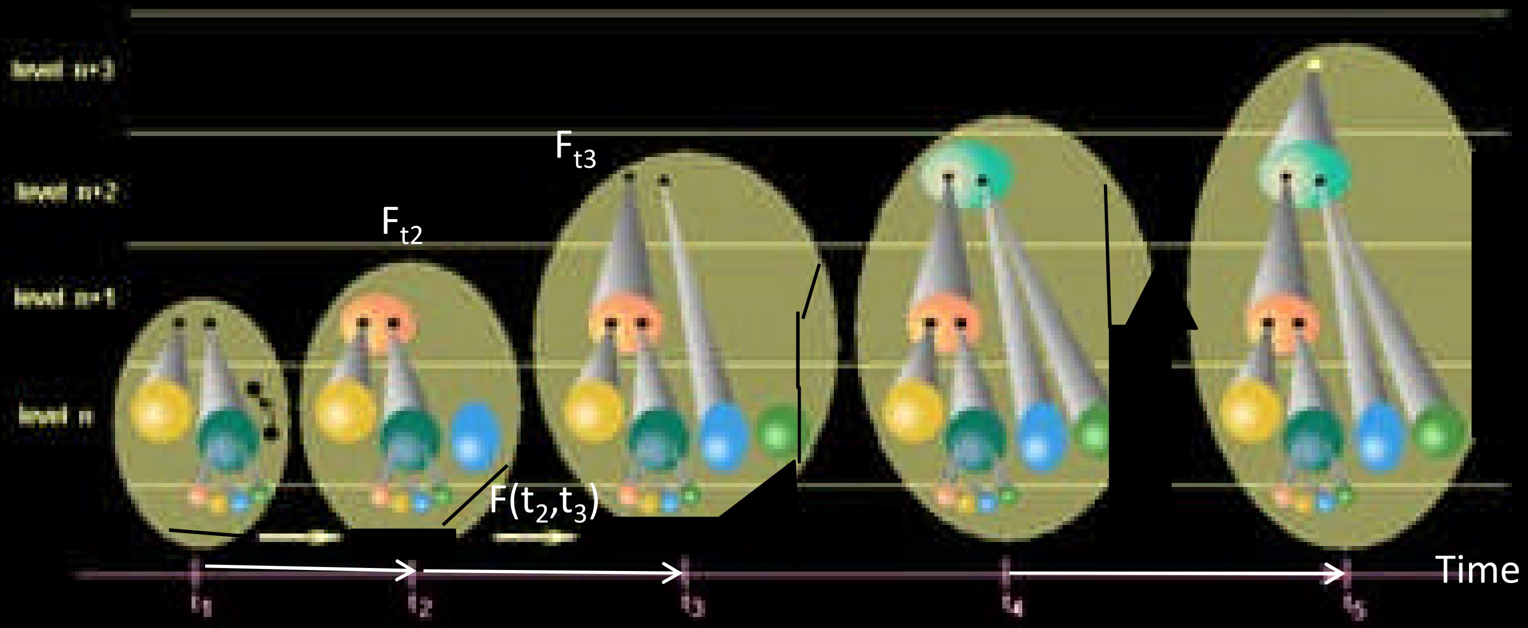
4.2. CONTROL SYSTEMS



A semi-action internal to (Top, open ins.) is called a *nucleus of action*; it defines a continuous semi-fibration **ρ: C*S** → **C**. Paired with a continuous functor **q**: **C** → **C'** it becomes a *Control System* (1963). A *solution* of this system is a continuous functor **σ: C'** → **C*S** which is a section of **qρ**. *Differentiable Control Systems* are obtained by replacing Top by Diff. The primitive example models the solutions of differential equations depending on a parameter.

The main results on Control Systems (1963-66) develop a categorical frame for studying Cauchy boundary problems (with applications to Burmister's equation in elasticity), and for obtainaing optimization theorems for solutions of a control system (possibly in terms of distributions), which specify the dynamic programing method of Bellman

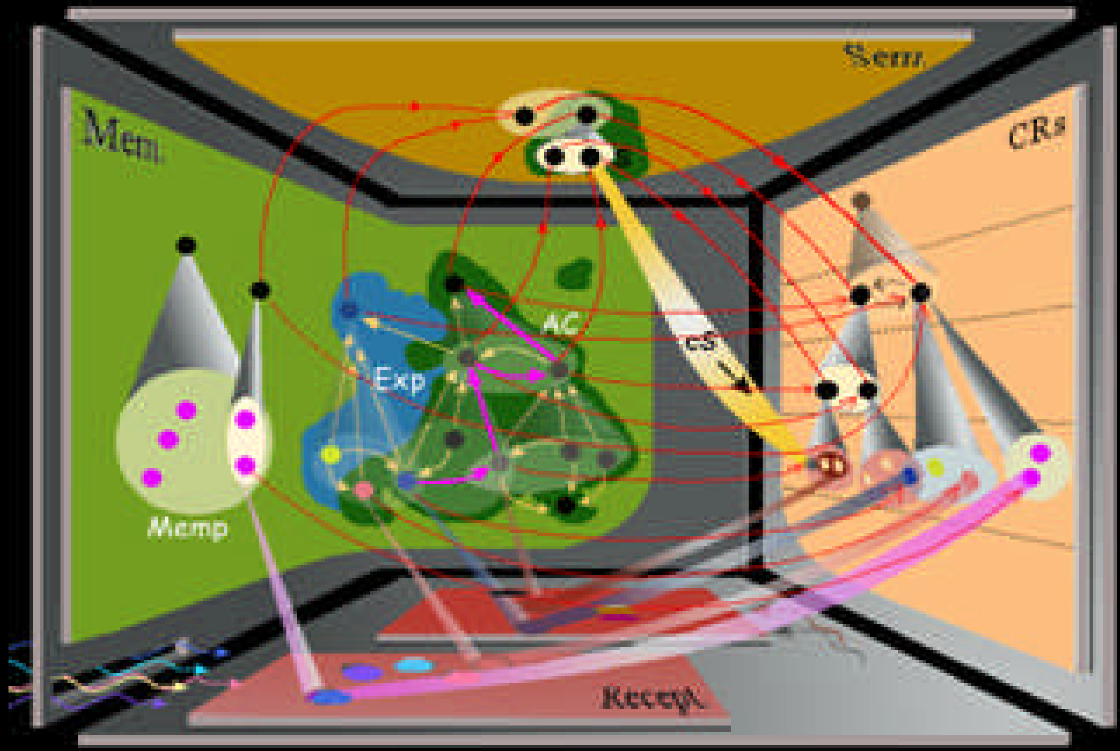
4.3. HIERARCHICAL EVOLUTIVE SYSTEMS



The *Memory Evolutive Systems* have been introduced (with J.-P. Vanbremeersch, 1986-2007) to model living systems. An *Evolutive System* is a semi-sheaf F of categories on the category $Time^{op}$, where Time defines the order on a subset of $\mathbf{R}_+$. The fiber F_t on t models the components of the system and their relations around t, and $F(t_i, t_j)$ the change from t_i to t_j. This 'transition' generally corresponds to a completion process (or '*complexification*') to add and/or suppress particular limits and/or colimits.

A *hierarchical category* is a category whose objects are partitioned in a finite number of levels so that an object of level n is the colimit of at least one functor toward the sub-category whose objects are of level < n. Let HCat the category of hierarchical categories with functors preserving the levels. A *Hierarchical Evolutive System* is a (HCat, Ins.)-semi-sheaf on $Time^{op}$,

4.4. THE MODEL MENS FOR COGNITIVE SYSTEMS



The *Memory Evolutive Systems* modeling living systems are HES, with sub-systems, the coregulators CRs and the Mem(ory), allowing for an internal regulation.

From the ES of neurons modelling the neural system of an animal, successive complexifications lead to the model MENS (Memory Evolutive Neural System) of a cognitive system (2001) in which a procedural and a semantic memory Sem develop. We model the formation of an interconnected personal memory, the *archetypal core AC* and the process by which it allows the formation of an internal global landscape at the basis of conscious processes.

Fore more details on MES,