EVOLUTIVE SYSTEMS IN MATHEMATICS

by

Andrée C. EHRESMANN
Université de Picardie Jules Verne
ehres@u-picardie.fr
http://pagesperso-orange.fr/ehres

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IN 1973

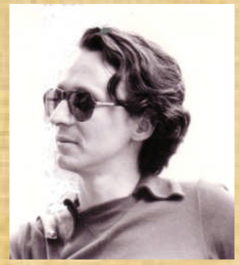
In Saint-Valery (near Amiens) during the 1st Conference on the Algebra of Categories, Amiens 1973

CHANTILLY 1975

A collective photo taken during a visit to Chantilly on the last day of the 2nd Conference on the Algebra of Categories, Amiens July 1975

SOME LANDMARKS

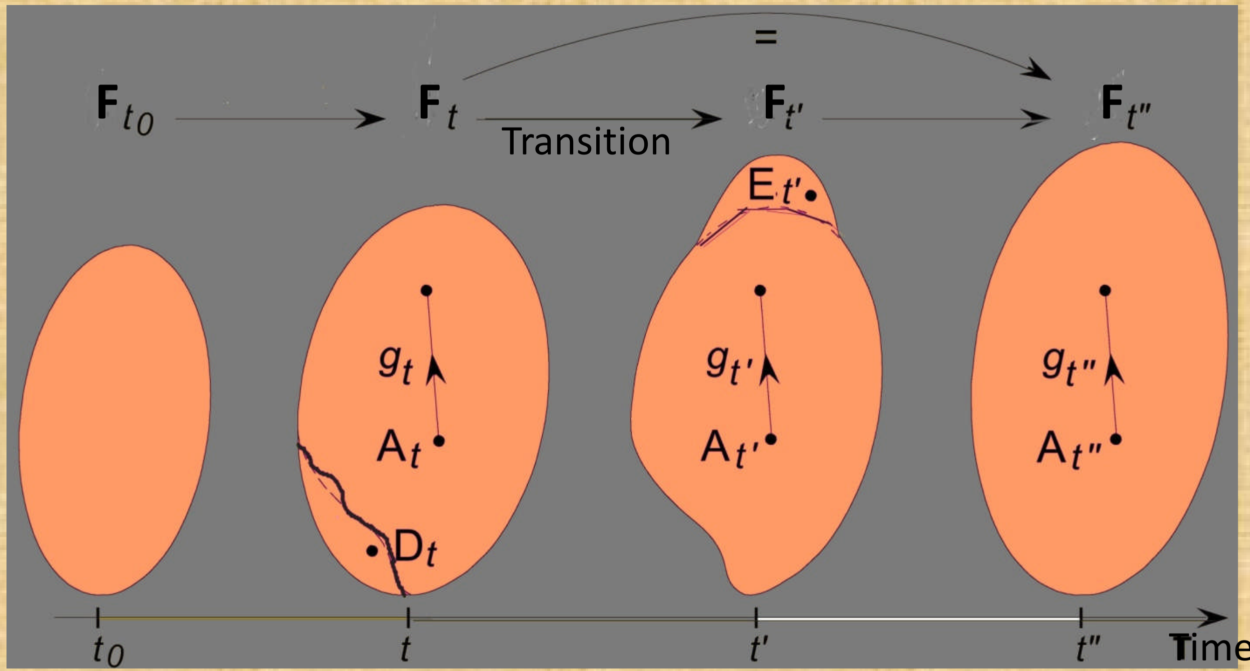
"Mémoire" with Mersch on a paper of Charles on completions. Thesis (1972) directed by Lavendhomme on enriched algebra, where he introduces the notion of an indexed limit. He exposed it in our Paris seminar in1972.
Assistant in Leuven (1970-73), Chargé de cours, then Professor in Louvain-la-Neuve, where he supervises twelve theses. Dean from 1996 to 2001.
Main actor in the diffusion of category research through the organization of several meetings and invitations to Louvain, his edition of books, and his 11 comprehensive books.



Prepares his "thèse de 3e cycle" (1973) on internal monads in our research team "TAC",
Assistant in Lille in 1973, Maître de Conférences in Amiens from 1977 to 1994. In 1984 I untrust him with the direction of "TAC".
"Doctorat d'Etat" in 1990 with Jean Benabou, on his 4 main papers on cohomology.
Professor in Calais since 1993. There he creates the pure mathematics laboratory. He has organized several SIC meetings and, in June, the 2008 International Conference.

Both have multiple collaborations

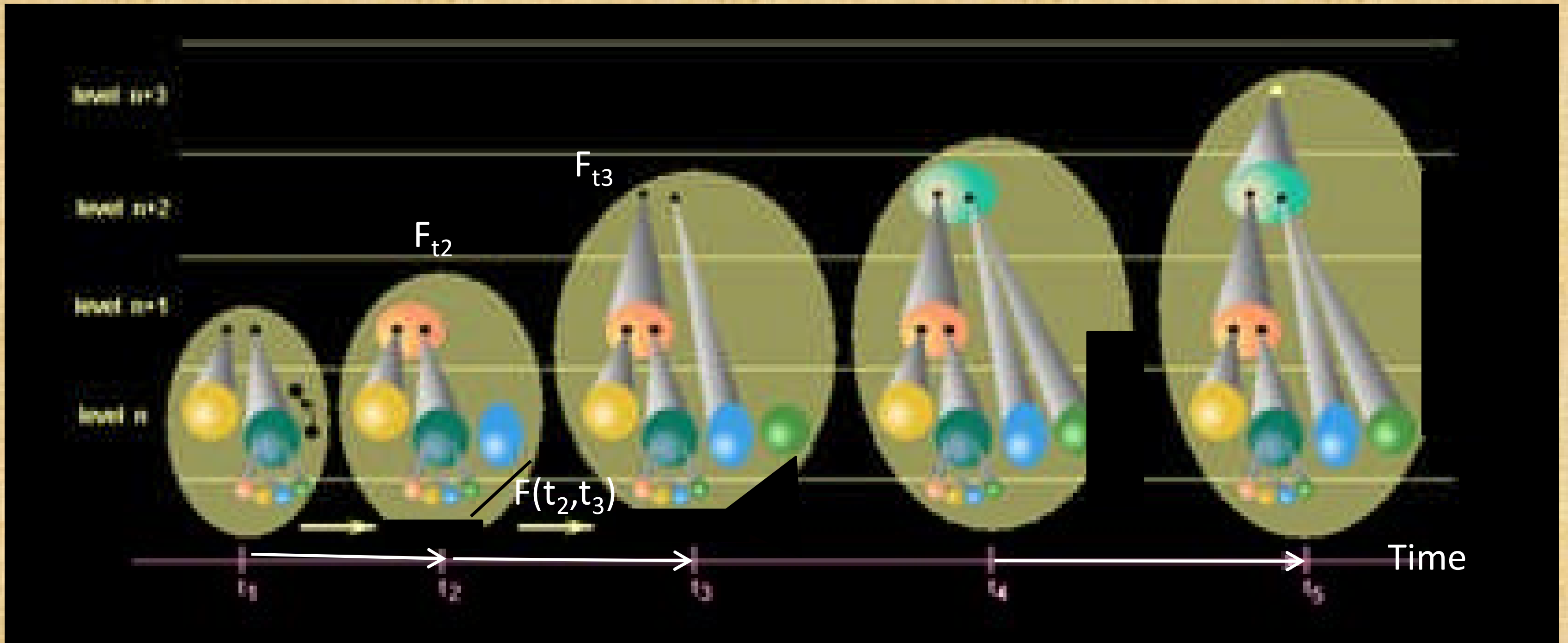
EVOLUTIVE SYSTEMS



The *Memory Evolutive Systems* have been introduced (with J.-P. Vanbremeersch, 1986-2007) to model living systems. Here the idea is to apply the MES philosophy to a MES, *Math,* describing the mathematical community with its individuals, the groups they form, and their 'memory' (papers, books,...)

An *Evolutive System* (ES) is a semi-sheaf of categories over a category *Time* defining the order on a finite or connected subset of R^{+}, called its *timescale*. Thus it associates to each t a category $\mathbf{F}_t$ and to $t' > t$ a functor 'Transition' from a sub-category of $\mathbf{F}_t$ to $\mathbf{F}_{t'}$ sending A_t on its 'new' state $A_{t'}$ at t' if it exists, and so that: If $A_{t'}$ is defined, then A_t has a state $A_{t''}$ for $t'' > t'$ if and only if $A_{t'}$ admits $A_{t''}$ as a state at t''. A component of the system (e.g. an individual) is modeled by the sequence (A_t) of its successive states.

HIERARCHICAL EVOLUTIVE SYSTEMS

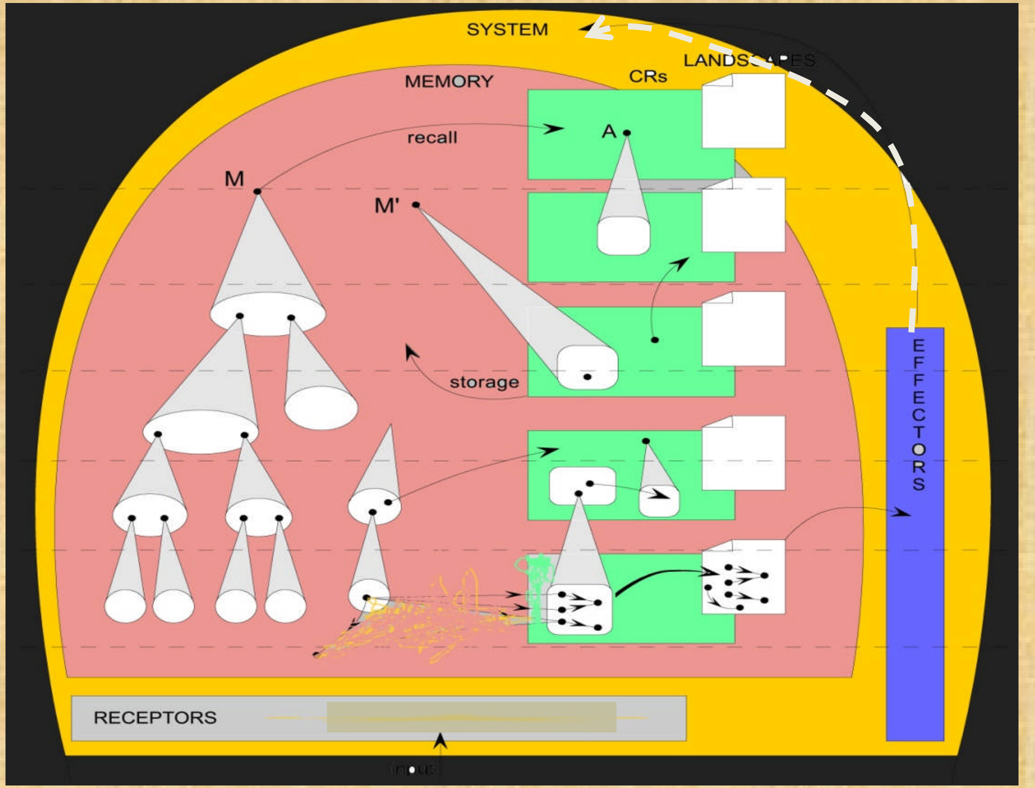


A *hierarchical category* is a category whose objects are partitioned into a finite number of levels so that an object of level n is the colimit of at least one diagram into the sub-category whose objects are of level $< n$.

A *Hierarchical Evolutive System (HES)* is an ES in which the $\mathbf{F}_t$ are hierarchical and the transitions preserve the levels. We say that $\mathbf{F}_{t'}$ is a *complexification* of $\mathbf{F}_t$ for a *strategy* S if S defines a sketch on a surcategory of $\mathbf{F}_t$ and if the transition $\mathbf{F}(t, t')$ is the functor from $\mathbf{F}_t$ to the prototype of this sketch (so that this transition has for effect to add/suppress the limit or colimit of some distinguished cones).

In *Math* the individuals (level 0) are organized in small groups (e.g., research teams), then departments, universities,..., Over time, some groups decline, others are formed.

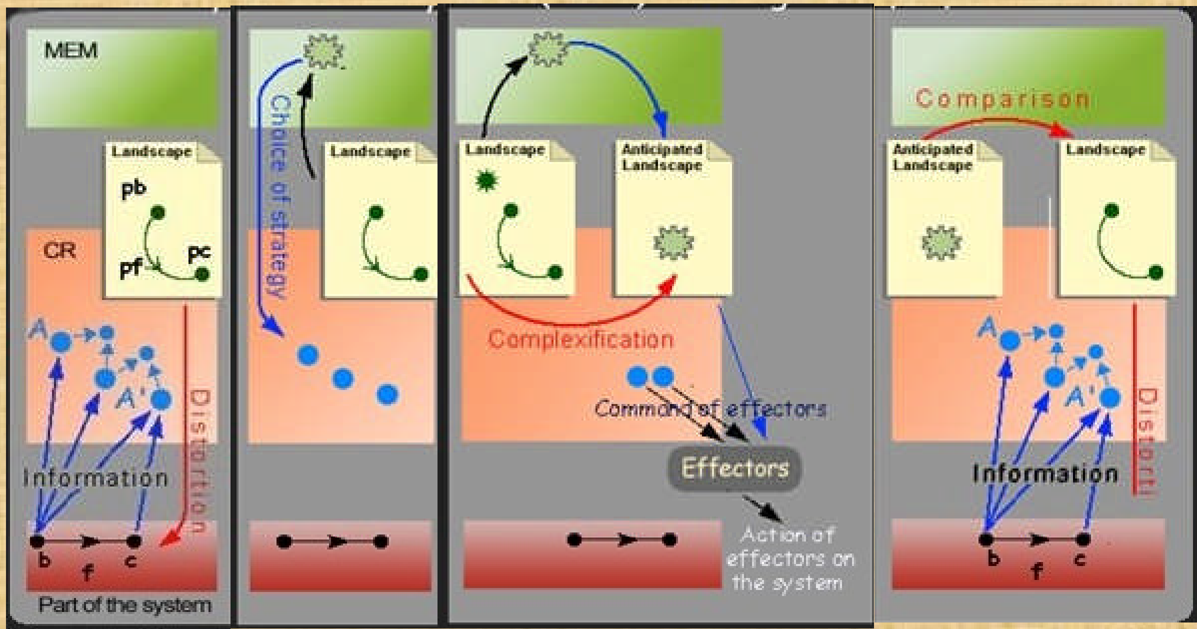
MEMORY EVOLUTIVE SYSTEMS



A *Memory Evolutive* System is a HES with a net of sub-ES of various levels, the Coregulators, which model specialized internal regulation organs, and a sub-HES, the Memory, related to a learning process. A morphism of a MES has a 'propagation delay' (additive) and a 'weight' (multiplicative) varying over time. Each CR has its own discrete timescale defining its *time-lag p, a threshold h* and a differential access to a part of the memory whose objects model its admissible 'strategies'.

In *Math* the CRs of increasing levels can be: an individual, a group of individuals working together, specialized research centers, universities, The memory contains books, varied archives...

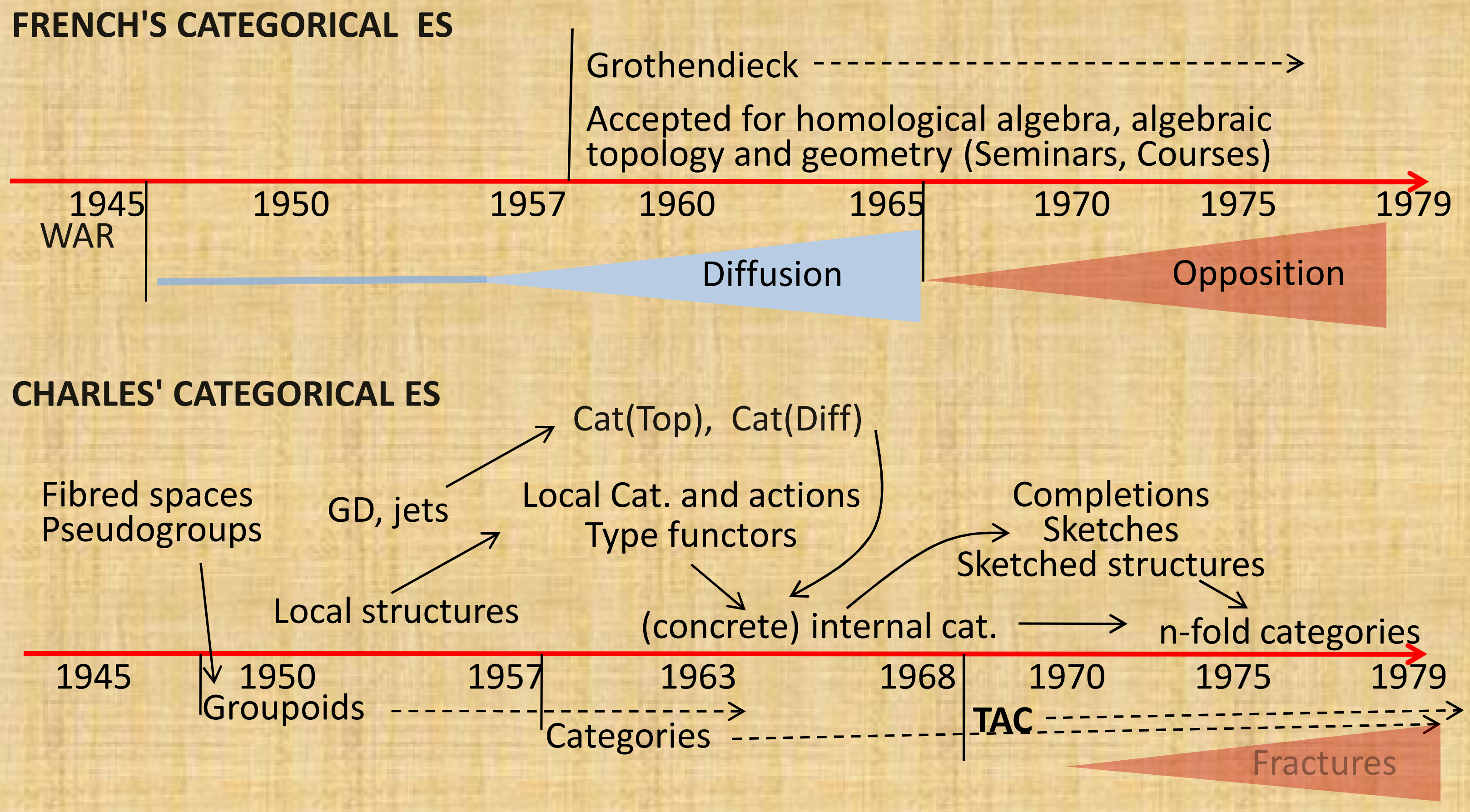
LANDSCAPE OF A CR



A step of a CR from *t* to *t'* has 3 phases: 1. Formation of its actual *landscape* L_t whose objects (or *perspectives*) are components of the subcategory of the comma category $F_t \downarrow CR$ formed of morphisms with a propagation delay $<p$ and a weight $>h$. 2. Selection of an admissible strategy S on L_t. 3. Carrying out of S modelled by the complexification L' of L_t for S. At the next step, comparison of L' with the new $L_{t'}$.
At a given time, there is an 'interplay' among the commands of the strategies of the various CRs, and the operative strategy carried out on the system is its result. It may cause a *fracture* to the CRs whose strategies cannot be achieved.

For a research team, the landscape gathers the information received by its members, the strategies correspond to collective tasks (e.g. planifying a conference), the effectuation to carry them out (its organisation) there is a fracture if it is not possible,

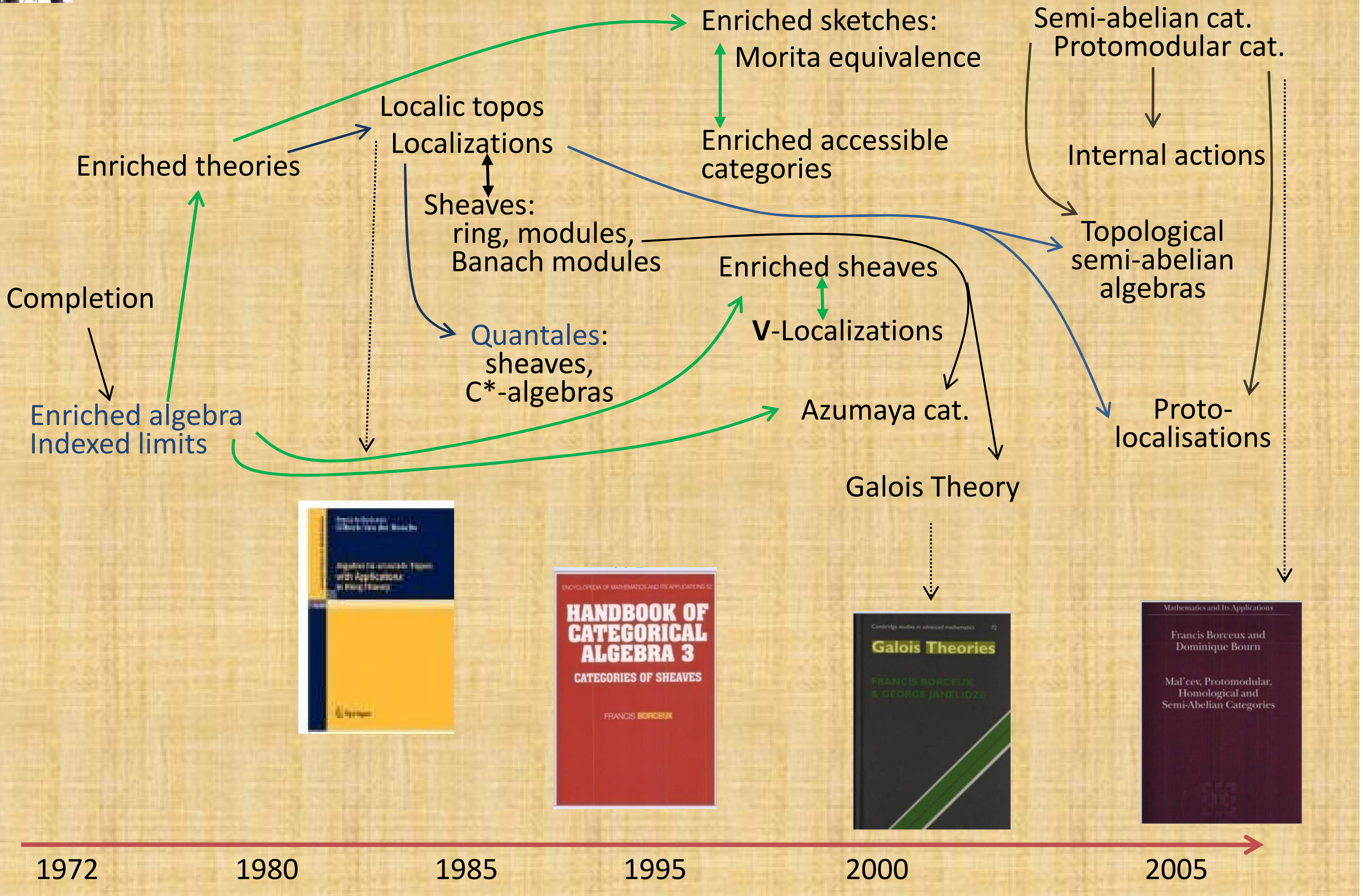
CATEGORIES IN FRANCE

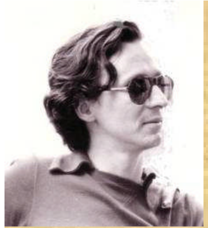


Up to the late 50's categories were ignored by most mathematicians (e.g., Choquet). At the same time, fierce opposition to probabllity theory and logic.

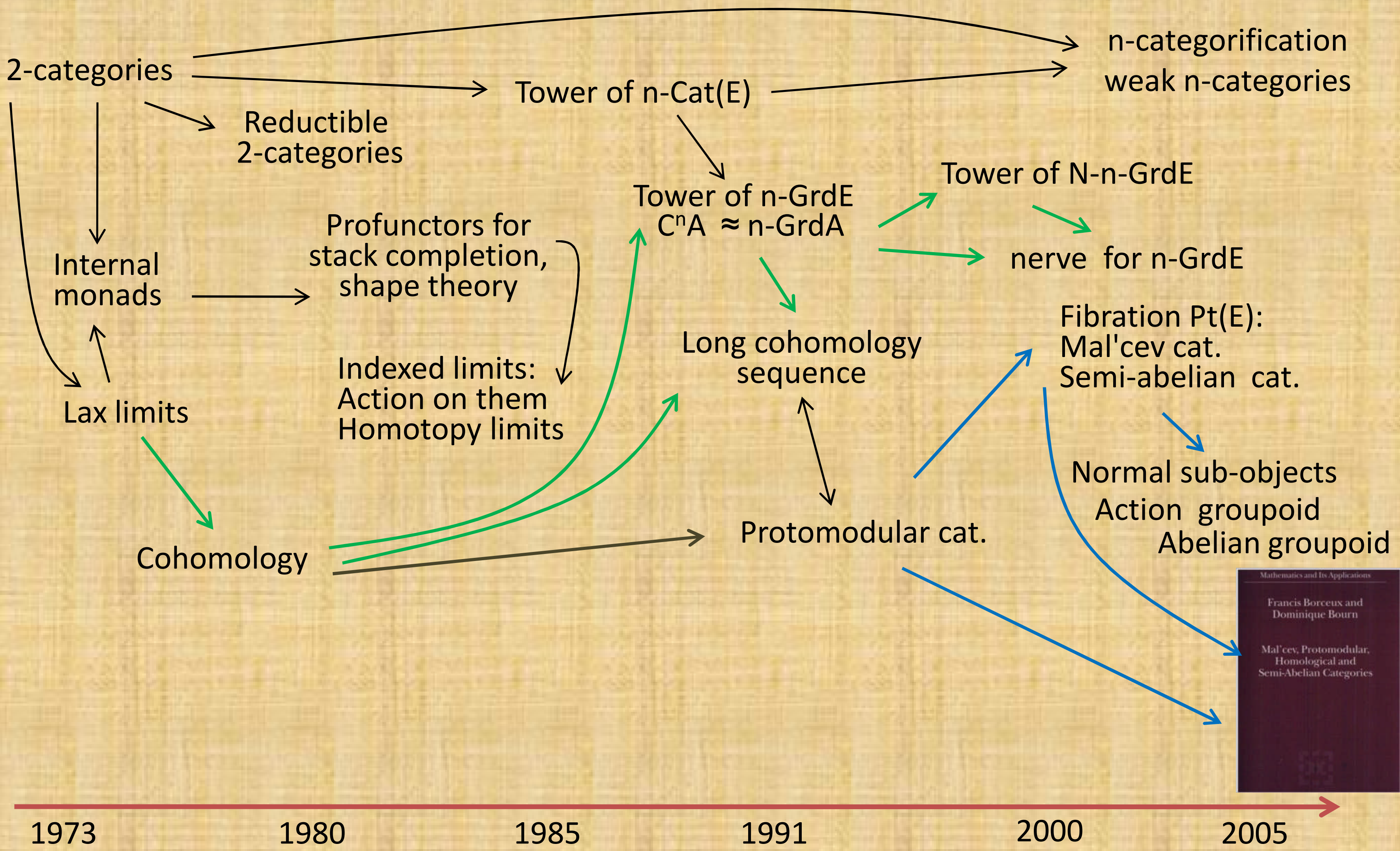
Charles used groupoids in the late 40's, but general categories only from 1957 on, first motivated by differential geometry and the definition of mathematical structures, e.g. local structures, then for their themselves. Our research team TAC (Paris-Amiens) was officially recognized in 1968, and more and more opposed in the seventies.

FRANCIS' EVOLUTIVE SYSTEM





DOMINIQUE'S EVOLUTIVE SYSTEM



NON-ABELIAN COHOMOLOGY

In 1964: Charles had generalized to a functor F: C → Grd the notion of crossed homomorphism: it is a section of the associated fibration; He defines $H^0(C,F)$ and $H^1(C,F)$ as the sets of components of the limit and of the lax limit of F.

1973. As an application of his theory of lax limits, DB extends this to H^2: If C is a monoid and F: C → Ab, then $H^2(C, F)$ = set of components of the 2-lax limit of F. It motivates him to define internally n-categories and n-limits to obtain a similar result for H^n.

1986. **Tower of exact fibrations,** if E is a Barr exact category:

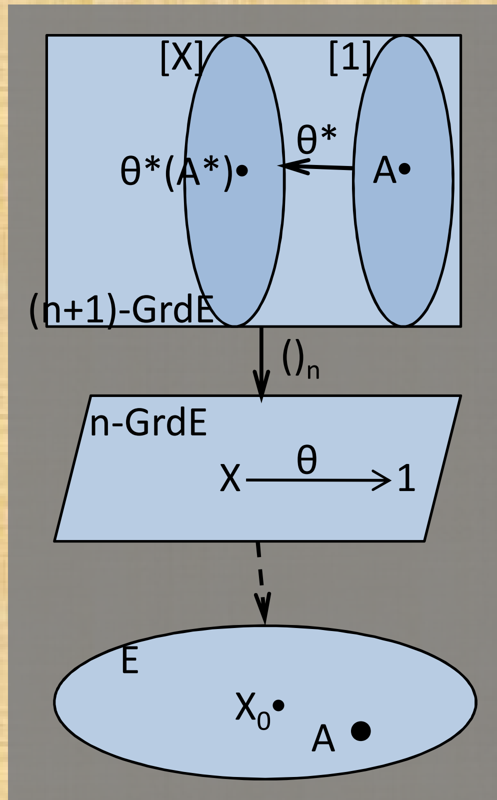
$$1 \leftarrow E \leftarrow \text{GrdE} \leftarrow \text{2-GrdE} \leftarrow \ldots \leftarrow \text{n-GrdE} \xleftarrow[()_n]{} \text{(n+1)-GrdE} \leftarrow \ldots$$

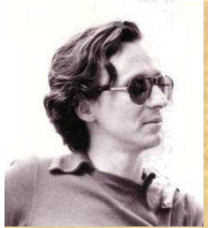
Theorem (generalizes Dold-Kan). If E is additive, the category n-GrdE of internal n-groupoids in E is equivalent to the category C^nE of positive complexes of length n.
There are n different monoidal closed structures on n-GrdAb.

1989. **Long cohomology sequence.** It is obtained if E is Barr exact and A an abelian group object in E, by defining the group $H^n(E, A)$ by induction : $H^0(E, A)$ = group of global elements of A,

$$H^{n+1}(E, A) = \text{colim}_X H^n([X], \theta^*(A^*)),$$

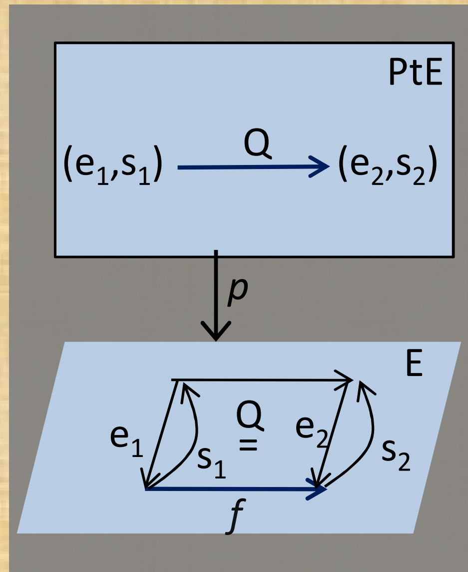
the colimit being taken on all the X with global support θ: X → 1, where [X] = fibre on X of $()_n$ and A* the abelian group object in [1] defined by A. If E is abelian, then H^n gives back Ext^n.





FUNCTOR NERVE. CATEGORIFICATION

1987. The *fibration of pointed objects p:* PtE → E has for objects the pairs (e_i, s_i) of an epi e_i of E and a splitting s_i, for morphisms the commutative squares Q between them; and p(Q) = f.



Theorem. If E is left exact, GrdE is monadic over PtE. Whence an *enriched comprehension scheme* if E is Barr exact.

1998. **Tower of normalized n-groupoids**: By induction, an n-groupoid X_n in E is *normalized* if X_{n-1} = $()_{n-1}(X_n)$ is normalized and if $t_n : X_n \to 1$ in the fibre $[X_{n-1}]$ splits. Then there is a tower of fibrations:

E ← PtE ← N-GrdE ← N-2-GrdE ← ... ← N-n-GrdE ← N-(n+1)-GrdE ← ...

where N-(n+1)-GrdE is monadic over N-n-GrdE, the corresponding category of algebras being (n+1)-GrdE.

2000. The functor *Nerve* connects the algebras of this tower with those of the tower of split n-simplicial objects in E.

2005. With J. Penon, construction of the *categorification* of structures defined by cartesian monads, through an iteration process. It is applied to construct *weak n-categories* and compare the limit of the iteration with Leinster weak ∞-categories.

ENRICHED ALGEBRA

V is a symmetric monoidal closed category.

1972;1975 (+Kelly). Definition and study of various enriched notions. In 72, introduction of the *indexed limit* L of (F, G), where F: **A**→ **V** and G: **A** → **B** are **V**-functors:

Nat(F, **B**(B, G-)) ≈ **B**(B, L) for each B.

1996 (+ Quinteiro). If **V** is locally finitely presentable + Barr regular, there are bijections:

localizations of [$\mathbf{C}^{op}$, **V**] ⟺ universal closure operations on it ⟺
V-Grothendieck topologies J on **C** ⟺ categories of **V**-*enriched sheaves* for J.

1998 (+Quinteiro, Rosicky). If **V** is locally finitely presentable, definition and study of *enriched accessible categories* and *enriched sketches;* a 'Morita Theorem' is proved.

Theorem, If **C** is a **V**-category:

C is **V**-accessible ⟺ **C** is **V**-sketchable.

The accessibility level can be defined by a class of colimits rather than a cardinal.

2002 (+Moens). Generalization of the theory of regular modules to **V**-*categories without units*, with applications in algebra and analysis, e.g., to Hilbert-Schmidt operators.

2002 (+Vitale). Theory of Azumaya **V**-categories and of the **V**-Brauer group. It extends the classical theory where **V** = $\mathbf{Mod}_R$ (R a commutative ring with unit) to the case of topological, matrix or Banach modules and to their sheaves,

ALGEBRAIC THEORIES. SHEAVES

1983. Study of the localizations of the category Sh(**H**,**T**) of sheaves of **T**-algebras on a locale **H**, if **T** is a finitary algebraic theory. Its 'formal initial segments' form the open sets of a compact space, the *pure spectrum* of **T**.

Sheaf Representation Theorem. A **T**-algebra is isomorphic to the **T**-algebra of global sections of a sheaf on the pure spectrum of **T**.

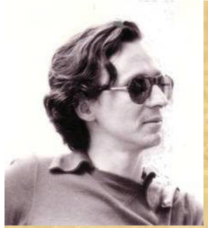
Applications are given to the theory of modules on a ring with unit and to the theory of Gelfand rings and their representations..

1991 +Pedicchio). **Characterisation Theorem.** The categories of separated objects for a Lawvere-Tierney topology on a Grothendieck topos are the locally presentable locally cartesian closed categories in which strong equivalence relations are effective.

1993 (+Cruciani, Berni-Canani), **Quantales** are multipicative lattices generalizing the lattice of closed right ideals of a non-commutative C*-algebra. Sheaves over a quantale Q are defined in 3 equivalent ways. Applications to C*-algebras are developed.

2005 (+Clementno). **Theorem.** If **T** is an algebraic theory such that $\mathrm{Set}^{\mathbf{T}}$ is semi-abelian, then the category $\mathrm{Top}^{\mathbf{T}}$ of its topological models is homological.

2008 (+Clementino, Gran, Sousa, Mantovani). Definition of a *protolocalisation* of a homological or semi-abelian category (the reflection preserving short exact sequences). It is associated to a closure operator and a torsion theory. Extension to regular categories. Examples in algebra and topos theory.

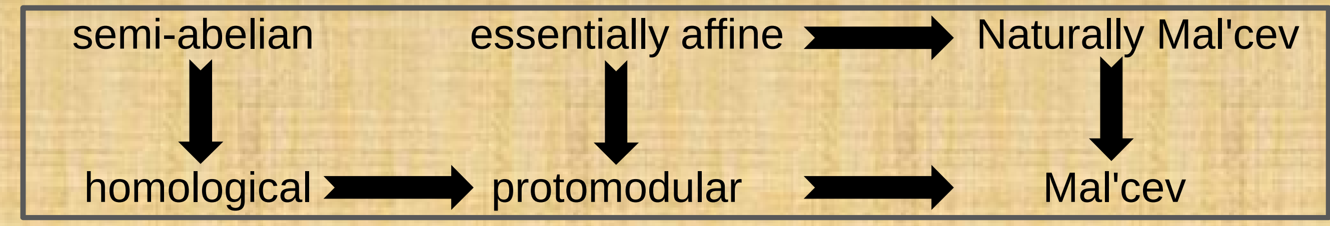


NON ADDITIVE (CO)HOMOLOGICAL ALGEBRA

In 1966, Charles tried to define a non-abelian cohomology by adapting the notion of exact sequence, e.g. in *Cat*. Now the approach is to characterize non-additive categories supporting (usual) exact sequences, homological lemmas and cohomology as in *Gp*. A main candidate seems to be the protomodular categories introduced by DB.

1991. A category E is *protomodular* if it is left exact and if, in the fibration of pointed objects *p*: PtE → E, each change of base reflects isomorphisms. Then an internal category in E is a groupoid. Protomodularity is stable by slices and coslices. Many examples, e.g. each fibre of the fibration $()_0$: GrdC → C of a left exact category C.

1996. *p* allows a comparison with other near-by notions: E is *naturally Mal'cev* iff *p* is additive; *Mal'cev* iff *p* is unital; *essentially affine* iff *p* is trivial, *additive* iff it has a 0 and *p* is trivial; if it is *semi-abelian,* the changes of base are monadic. Whence:



1999. Notions of *normal subobject* and *abelian object* in a protomodular category.

2007. The joint book (opposite) and recent papers study these various kinds of categories and their homological properties. Notions such as central morphisms, action groupoids and abelian groupoids are analysed.